Two

I. The only other rookwise-connected antimagic square is the "complement" of the one in Figure 2. Simply change each digit to the difference between that digit and 10. The result is a square that can be obtained by spiraling the digits in the same way as before, but taking them in reverse order:

987
216
345

One way to prove there are no other such squares starts by observing that if the matrix is colored like a chessboard, with white cells at the corners, the odd digits must be on white, even digits on black. Digits 2 and 4 cannot be opposite because 3 would have to go between, and this makes a rook path impossible. Either 8 or 6, therefore, must be opposite 2. The path must start and end on white. It takes only a few minutes to check the four essentially different patterns for duplication of sums.

I had not seen an antimagic square before Dr. Matrix introduced me to them. The earliest example I later found of such a square is the order-3 square given in *Sam Loyd and His Puzzles* (1928) as the answer to a puzzle on page 44.

In *Mathematics Magazine* (Jan. 1951) Dewey Duncan defined a heterosquare as a square in which no two rows, columns, or diagonals (including "broken diagonals") have the same sum. (The order-3 square has four broken diagonals. Referring to the square shown in Figure 2, they are the cells bearing the number triplets 1, 6, 4; 8, 2, 5; 3, 8, 6; and 2, 4, 7. Thus the antimagic square in Figure 2 is not a heterosquare; for one thing, the third broken diagonal adds up to 17, and so does the second column.) Duncan asked for a heterosquare of order-3 and proof that no such square of order-2 exists. It is easy to show that an order-2 is impossible. A proof that the order-3 also is impossible was given by Charles F. Pinzka in *Mathematics Magazine* (Sept.-Oct. 1965, pp. 250–252). Order-4 squares *are* possible; Pinzka gave two. Another proof of impossibility for the order-3 was given by Prasert Na Nagara in the same magazine (Sept.-Oct. 1966, pp. 255–256). Nagara also found two "almost" heterosquares of order-3 in which all sums but two were distinct.

J. A. Lindon, writing in *Recreational Mathematics Magazine* (Feb. 1962), proposed searching for antimagic squares in which the sums of the rows, columns, and main diagonals (broken diagonals not considered) are not only different but form a sequence of consecutive integers. A summary of Lindon's results, with some new material added, appears in Joseph Madachy's *Mathematics on Vacation* (New York: Scribner, 1966), pp. 101–110. No order-2 square of this type is possible. Order-3 also is impossible, although one can come close, as the following square (from C. C. Verbeek's *Puzzel met Plezier*, Amsterdam, 1962, p. 155) shows:

268
791
534

All eight sums are distinct, and only one diagonal sum, 22, is outside the sequence.

Many order-4 and higher antimagic squares, with all sums in consecutive order, were found by Lindon.

Charles W. Trigg, writing on "The Sums of Third Order Anti-Magic Squares," *Journal of Recreational Mathematics* 2 (1969): 250–254, showed that the eight sums of an order-3 antimagic square cannot be in any arithmetic progression, thus confirming Lindon's conjecture that they cannot be consecutive. He also proved that the eight sums cannot all be even.

In a note on "A Remarkable Group of Antimagic Squares," *Mathematics Magazine* 44 (1971): 13, Trigg examined the eight patterns obtained by placing 1 in the center of the three-by-three array, the sequence 3, 5, 7, 9 in the corners, and the sequence 2, 4, 6, 8 in the side cells. "Remarkably, whether the sequences run clockwise or counterclockwise, each of the eight essentially distinct squares thus obtained is antimagic."

The complements of these eight squares are also antimagic. When the four broken diagonals are considered, it turns out that each of the sixteen squares is also "almost heterosquare in having only two duplicate sums. Commenting on problem 84 in the same magazine, 4 (1971):236–237. Trigg has given a method that produces 108 order-3 arrays that are almost heterosquare. The total number of distinct order-3 antimagic squares, and the number of distinct order-3 almost heterosquares, remain unknown.

II. The equation asked for is:

$$36^2 + 37^2 + 38^2 + 39^2 + 40^2 = 41^2 + 42^2 + 43^2 + 44^2$$

I am indebted to Russell L. Linton, Oakland, California, for pointing out in a letter that the first integer in the series of such equations is obtained by the formula $n(2n+1)$, where n is the number of terms on the right side of the equation. Thus, to write the next example, which has five terms on the right, we substitute 5 for n to obtain $5(10+1)=55$. We can immediately write:

$$55^2+56^2+57^2+58^2+59^2+60^2=61^2+62^2+63^2+64^2+65^2$$

A discussion of this series, "Runs of Squares," by T. H. Beldon, appeared in the *Mathematical Gazette* (Dec. 1961, pp. 334–335).

The series has a trivial analogy with the following first-power series:

$$1+2=3$$
$$4+5+6=7+8$$
$$9+10+11+12=13+14+15$$

III. The three-symbol chain problem has a fascinating history that begins with a two-symbol chain first discovered by the Norwegian mathematician Axel Thue and described by him in 1912. Begin with 01. For the 0, substitute 01, and for the 1, substitute 10. The result is a chain of four digits: 0110. Repeating this procedure, changing each 0 to 01 and each 1 to 10, produces the chain 01101001. In this way we can form a chain as long as we wish, each step doubling the number of digits and forming a chain that starts by repeating the previous chain. This sequence of symbols, called the Thue series, has the remarkable property that no block of one or more digits ever appears three times consecutively. The chain may "stutter" once, but whenever this occurs, regardless of the size of the block that repeats, the very next digit is sure to be the wrong one for a third appearance of the block.

Max Euwe, a former world chess champion, was among the first to recognize that the Thue sequence provides a method of playing an infinitely long game of chess. The so-called German rule for preventing such games declares a game drawn if a player plays any finite sequence of moves three times in succession in the same position. Two players need only create a position in which each can move either of two pieces back and forth, regardless of how the other player moves his two pieces. If each now plays his two pieces in a Thue sequence, neither will ever repeat a pattern of moves three times consecutively.

From the Thue series it is easy to derive a three-symbol chain that solves Dr. Matrix's problem. First, we transform it to a chain of four symbols by writing 0 under every 00 pair, 1 under every 01 pair, 2 under every 10 pair, and 3 under every 11 pair:

Thue series:	0 1 1 0 1 0 0 1 . . .
Four-symbol chain:	1 3 2 1 2 0 1 . . .

This infinite four-symbol chain has the property that no finite block of digits ever appears *twice* side by side. It can now be transformed to a three-symbol chain, with the same property, by replacing every 3 with a 0:

Four-symbol chain:	1 3 2 1 2 0 1 . . .
Three-symbol chain:	1 0 2 1 2 0 1 . . .

This solution to the three-symbol problem was given by Marston Morse and Gustav Hedlund in an important 1944 paper, "Unending Chess, Symbolic Dynamics and a Problem in Semigroups," *Duke Mathematics Journal* 11 (1944):1–7. There were earlier solutions (including one by the Russian mathematician S. Arshon in 1937) and many later ones. John Leech gave this solution in "A Problem on Strings of Beads," *Mathematical Gazette* 41 (1957):277–278:

Consider the following three blocks of digits:

0 1 2 1 0 2 1 2 0 1 2 1 0

1 2 0 2 1 0 2 0 1 2 0 2 1

2 0 1 0 2 1 0 1 2 0 1 0 2

The digits in these blocks are so arranged that if we substitute the three blocks for the three digits (replacing 1 with one block, 2 with another, 3 with the third) in any stutter-free chain (e.g., any one of the three blocks), the resulting chain will also be stutter-free. In this longer chain we can now substitute blocks for digits once more to obtain a still longer chain, and so on ad infinitum.

It is not possible to construct shorter palindromic blocks (blocks that are the same backward as forward) that can be used in this way, but shorter asymmetric blocks are possible. Allan Beek sent me a similar solution using the following asymmetric blocks of eleven digits each:

12313231213
12321312132
12321323132

It is not known if there is a set of three shorter blocks that provides a proof of this type.

The three-symbol chain furnishes a way of evading the rule for drawn chess games even if the rule is strengthened by declaring a game drawn if a finite sequence of moves occurs only twice in succession. Each player simply moves three pieces in a pattern given by the three-symbol chain.

There are other ways of generating the Thue series than the one explained above. In 1961 Dana Scott sent me the following. First write the sequence of integers in binary form: 0, 1, 10, 11, 100, 101, 110, 111, 1000 . . . Next replace each number with 1 if it contains an odd number of 1's, and with 0 if it contains an even number of 1's. The result, surprisingly, is the Thue series: 011010011 . . .

A method of transforming the Thue series directly to a three-symbol solution of Dr. Matrix's problem was explained in 1963 by C. H. Braunholtz, "An Infinite Sequence of Three Symbols with No Adjacent Repeats," *American Mathematical Monthly* 70 (1963):675–676. In the Thue series the number of 1's between any 0 and the next 0 is either 0, 1, or 2. There are two 1's between the first and second 0, one 1 between the second and third 0's, none between the third and fourth, and so on. The numbers of these 1's, as we proceed from 0 to 0, form a three-symbol infinite series, 2102012 . . . , with the required property.

P. Erdös proposed the following three-symbol chain problem that is the same as the one given by Dr. Matrix except that two blocks of digits are now considered "identical" if each symbol appears in them the same number of times. For example, 00122 = 02102 because each contains two 0's, one 1, and two 2's. The largest possible sequence that does not have two "identical" blocks side by side is one of seven digits, e.g., 0102010. It is not yet known if there is an infinite four-symbol chain with this property.

Other references on the Thue series and the three-symbol problem include:

Marston Morse, "A Solution of the Problem of Infinite Play in Chess," Abstract 360, *Bulletin of the American Mathematical Society* 44 (1938):632.

D. Hawkins and W. E. Mientka, "On Sequences Which Contain No Repetitions," *Mathematics Student* 24 (1956):185–187.

G. A. Hedlund and W. H. Gottschalk, "A Characterization of the Morse Minimal Set," *Proceedings of the American Mathematical Society* 15 (1964):70–74.

Richard A. Dean, "A Sequence Without Repeats," *American Mathematical Monthly* 72 (1965):383–385.

P. A. B. Pleasants, "Non-Repetitive Sequences," *Proceedings of the Cambridge Philosophical Society* 68 (1970):267–274.

T. C. Brown, "Is There a Sequence of Four Symbols in Which No Two Adjacent Segments Are Permutations of One Another?" *American Mathematical Monthly* 78 (1971):886–888.

R. C. Entringer, D. E. Jackson, and J. A. Schatz, "On Nonrepetitive Sequences," *Journal of Combinatorial Theory,* Series A, 16 (1974):159–164.

IV. The number 102564 quadruples in size if the 4 is moved from the back to the front, 410256; therefore, Miss Toshiyori's telephone number is 1–0256. Puzzles of this type are easily solved by a kind of multiply-as-you-go technique explained in Figure 37.

After mastering this method, readers may wish to tackle the following three problems:

1. What is the smallest number ending in 6 that becomes six times as large when the 6 is shifted from the end to the front? (Warning: The number has 58 digits!)

2. Find the smallest number *beginning* with 2 that triples when the 2 is moved to the end.

3. Prove that there is no number beginning with the digit *n* that increases *n* times when the first digit is moved from the front to the end, except in the trivial case where *n* is 1.

Readers interested in further explorations of problems of this type, in which digits are moved from one end of a