Chapter 11

The Prouhet–Tarry–Escott Problem

A classical problem in Diophantine equations that occurs in many guises is the Prouhet–Tarry–Escott problem. This is the problem of finding two distinct lists (repeats are allowed) of integers $[\alpha_1, \dots, \alpha_n]$ and $[\beta_1, \dots, \beta_n]$ such that

$$\begin{aligned} \alpha_1 + \cdots + \alpha_n &= \beta_1 + \cdots + \beta_n \\ \alpha_1^2 + \cdots + \alpha_n^2 &= \beta_1^2 + \cdots + \beta_n^2 \\ \vdots \quad & \quad \vdots \quad \vdots \\ \alpha_1^k + \cdots + \alpha_n^k &= \beta_1^k + \cdots + \beta_n^k. \end{aligned}$$

We will call this the Prouhet–Tarry–Escott Problem. We call n the size of the solution and k the degree. We abbreviate the above system by writing $[\alpha_i] =_k [\beta_i]$.

This problem has a long history and is, in some form, over 200 years old. In 1750–1751 Euler and Goldbach noted that

$$[a, b, c, a+b+c] =_2 [a+b, a+c, b+c].$$

A general solution of the problem for all degrees, but large sizes, came a century later, in 1851, when Prouhet found that there are n^{k+1} numbers separable into n sets such that each pair of sets forms a solution of degree k and size n^k. (See Theorem 1 of Chapter 12.) Prouhet's result, while the first general solution of the problem, was not properly noticed until Wright [1959] took exception to the problem being called the Tarry–Escott problem and drew attention to Prouhet's contribution in a paper called "Prouhet's 1851 solution of the Tarry–Escott problem of 1910." More of the early history of the problem can be found in Dickson [1952], where he refers to it as the problem of "equal sums of like powers."

The Diophantine equation above can be reformulated as a question about polynomials in two ways.

Theorem 1. *The following are equivalent:*

(a) $\sum_{i=1}^{n} \alpha_i^j = \sum_{i=1}^{n} \beta_i^j$ *for* $j = 1, \ldots, k-1$.

(b) $\deg\left(\prod_{i=1}^{n}(z-\alpha_i) - \prod_{i=1}^{n}(z-\beta_i)\right) \leq n-k$.

(c) $(z-1)^k \,\Big|\, \sum_{i=1}^{n} z^{\alpha_i} - \sum_{i=1}^{n} z^{\beta_i}$.

It is the third form above that rephrases the Prouhet–Tarry–Escott problem as a question on the vanishing of low-height polynomials.

An *ideal solution* is one where the degree is 1 less than the size, which is the maximum possible. An *even ideal symmetric solution* of size n is of the form

$$[\pm\alpha_1, \ldots, \pm\alpha_{n/2}] =_{n-1} [\pm\beta_1, \ldots, \pm\beta_{n/2}]$$

and satisfies any of the following equivalent statements:

(a) $\sum_{i=1}^{n/2} \alpha_i^{2j} = \sum_{i=1}^{n/2} \beta_i^{2j}$ for $j = 1, \ldots, \frac{n-2}{2}$.

(b) $\prod_{i=1}^{n/2} \left(z^2 - \alpha_i^2\right) - \prod_{i=1}^{n/2} \left(z^2 - \beta_i^2\right) = C$ for some constant C.

(c) $(1-z)^n \,\Big|\, \sum_{i=1}^{n/2} \left(z^{\alpha_i} + z^{-\alpha_i}\right) - \sum_{i=1}^{n/2} \left(z^{\beta_i} + z^{-\beta_i}\right)$.

Note that the third form of an even symmetric solution gives rise to a real (cosine) polynomial on the boundary of the unit disk.

An *odd ideal symmetric solution* of size n and even degree $n-1$ is of the form

$$[\alpha_1, \ldots, \alpha_n] =_{n-1} [-\alpha_1, \ldots, -\alpha_n]$$

and satisfies any of the following equivalent statements:

(a) $\sum_{i=1}^{n} \alpha_i^j = 0$ for $j = 1, 3, 5, \ldots, n-2$.

(b) $\prod_{i=1}^{n}(z-\alpha_i) - \prod_{i=1}^{n}(z+\alpha_i) = C$ for some constant C.

(c) $(1-z)^n \,\Big|\, \sum_{i=1}^{n} z^{\alpha_i} - \sum_{i=1}^{n} z^{-\alpha_i}$.

In the third form above, an odd symmetric solution gives rise, on multiplication by i, to a real (sine) polynomial on the boundary of the unit disk.

There is a trivial transformation on solutions. Any linear transformation of a solution is a solution ($\alpha_i \mapsto A\alpha_i + B$ with A and B integers). Two such solutions are called *equivalent*.

The following is a list of ideal solutions for sizes 2 through 12, excluding 11 where no solution is known. For each size it includes the smallest known solution. Except for the case of size 4, the solutions are all symmetric. Exactly two inequivalent solutions of size 9 are known, and exactly one inequivalent solution of size 12 is known. For the rest of the known cases there are infinite parametric families of inequivalent solutions.

$$\begin{gathered}
[\pm 2] =_1 [\pm 1], \\
[-2,-1,3] =_2 [2,1,-3], \\
[-5,-1,2,6] =_3 [-4,-2,4,5], \\
[-8,-7,1,5,9] =_4 [8,7,-1,-5,-9], \\
[\pm 1, \pm 11, \pm 12] =_5 [\pm 4, \pm 9, \pm 13], \\
[-50,-38,-13,-7,24,33,51] =_6 [50,38,13,7,-24,-33,-51], \\
[\pm 5, \pm 14, \pm 23, \pm 24] =_7 [\pm 2, \pm 16, \pm 21, \pm 25], \\
[-98,-82,-58,-34,13,16,69,75,99] \\
=_8 [98,82,58,34,-13,-16,-69,-75,-99], \\
[174,148,132,50,8,-63,-119,-161,-169] \\
=_8 [-174,-148,-132,-50,-8,63,119,161,169], \\
[\pm 99, \pm 100, \pm 188, \pm 301, \pm 313] =_9 [\pm 71, \pm 131, \pm 180, \pm 307, \pm 308], \\
[\pm 103, \pm 189, \pm 366, \pm 452, \pm 515] =_9 [\pm 18, \pm 245, \pm 331, \pm 471, \pm 508], \\
[\pm 151, \pm 140, \pm 127, \pm 86, \pm 61, \pm 22] =_{11} [\pm 148, \pm 146, \pm 121, \pm 94, \pm 47, \pm 35].
\end{gathered}$$

The main problem of this section is the question of the size of minimal solutions of the Prouhet–Tarry–Escott problem and specifically whether or not ideal solutions exist:

P2. The Prouhet–Tarry–Escott Problem. *Find a polynomial with integer coefficients that is divisible by* $(z-1)^n$ *and has smallest possible length. (That is, minimize the sum of the absolute values of the coefficients.)*

Wright [1934] specifically conjectures that it is always possible to find ideal solutions. This has interesting consequences for the so-called easier Waring problem that is discussed in the next section. Heuristic arguments suggest that Wright's conjecture should be false. Counting arguments, as in the next section, give solutions of degree n and size $O(n^2)$, and it is tempting to speculate that this is essentially best possible. It is, however, intriguing that ideal solutions exist for as many n as they do.

Parametric Solutions

We now present parametric solutions of size 5, 6, 7, 8, and 10. The families of solutions of size 6, 8, and 10 are all symmetric, and immediately (on replacing t^2 by t) give infinite families of solutions of size 3, 4, and 5 where all the α_i are squares.

Size 5. The following is a one-parameter example of size 5:

$$\begin{aligned} F_5 := {} & \left(t+2m^2\right)(t-1)\left(t+2m^2-1\right)\left(t-2m^2+1-m\right)\left(t-2m^2+m+1\right) \\ & -\left(t-2m^2\right)(t+1)\left(t-2m^2+1\right)\left(t+2m^2-1+m\right)\left(t+2m^2-m-1\right). \end{aligned}$$

This expands to

$$F_5 := -4m^2(m-1)(2m+1)(2m-1)(m+1)\left(2m^2-1\right).$$

The fact that the expansion is independent of t proves, by the second criterion of Theorem 1 (with $z = t$), that the example is correct.

Size 6. The following is a simple two-parameter example of size 6:

$$\begin{aligned} F_6 := {} & \left(t^2-(2n+2m)^2\right)\left(t^2-(nm+n+m-3)^2\right)\left(t^2-(nm-n-m-3)^2\right) \\ & -\left(t^2-(2n-2m)^2\right)\left(t^2-(n-nm-m-3)^2\right)\left(t^2-(m-nm-n-3)^2\right). \end{aligned}$$

On expansion, one sees that

$$F_6 := -16nm(m-1)(m+3)(m-3)(m+1)(n-1)(n+3)(n-3)(n+1).$$

It is possible to solve for symmetric solutions of size 6. (See C3.) This gives the following three-parameter solution of size 6 (in nonsymmetric form):

$$\begin{aligned} & \left[\frac{2}{3}\frac{a_3^2-b_1^2-b_2^2-b_2b_1}{-b_1+a_3-b_2}, \frac{a_3b_1+a_3b_2-b_2b_1-b_2^2-b_1^2}{-b_1+a_3-b_2}, a_3, \frac{2}{3}\frac{a_3^2-b_1^2-b_2^2-b_2b_1}{-b_1+a_3-b_2}-a_3,\right. \\ & \qquad\qquad \left.\frac{2}{3}\frac{a_3^2-b_1^2-b_2^2-b_2b_1}{-b_1+a_3-b_2}-\frac{a_3b_1+a_3b_2-b_2b_1-b_2^2-b_1^2}{-b_1+a_3-b_2}, 0\right] \\ =_5 & \left[b_1, b_2, \frac{a_3^2+b_2b_1-a_3b_2-a_3b_1}{-b_1+a_3-b_2}, \frac{2}{3}\frac{a_3^2-b_1^2-b_2^2-b_2b_1}{-b_1+a_3-b_2}-\frac{a_3^2+b_2b_1-a_3b_2-a_3b_1}{-b_1+a_3-b_2},\right. \\ & \qquad\qquad \left.\frac{2}{3}\frac{a_3^2-b_1^2-b_2^2-b_2b_1}{-b_1+a_3-b_2}-b_2, \frac{2}{3}\frac{a_3^2-b_1^2-b_2^2-b_2b_1}{-b_1+a_3-b_2}-b_1\right]. \end{aligned}$$

Size 7. The following gives a parametric solution of size 7. This is homogeneous in j and k, so it is really a one-parameter solution. This is a much simplified version of a result of Gloden [1944]. He gives a four-parameter solution, but two of the parameters are extraneous. Chernick [1937] also gives such a family. Let

$$\begin{aligned} F_7 := {} & (t-R_1)(t-R_2)(t-R_3)(t-R_4)(t-R_5)(t-R_6)(t-R_7) \\ & -(t+R_1)(t+R_2)(t+R_3)(t+R_4)(t+R_5)(t+R_6)(t+R_7), \end{aligned}$$

where

$$\begin{aligned}
R_1 &:= -\left(-3j^2k+k^3+j^3\right)\left(j^2-kj+k^2\right),\\
R_2 &:= (j+k)(j-k)\left(j^2-3kj+k^2\right)j,\\
R_3 &:= (j-2k)\left(j^2+kj-k^2\right)kj,\\
R_4 &:= -(j-k)\left(j^2-kj-k^2\right)(-k+2j)k,\\
R_5 &:= -(j-k)\left(-2kj^3+j^4-j^2k^2+k^4\right),\\
R_6 &:= \left(j^4-4kj^3+j^2k^2+2k^3j-k^4\right)k,\\
R_7 &:= \left(j^4-4kj^3+5j^2k^2-k^4\right)j.
\end{aligned}$$

On expansion,

$$\begin{aligned}
F_7 = 2j^3k^3&(-k+2j)(j-2k)(j+k)\\
&\times\left(j^2+kj-k^2\right)\left(j^2-kj-k^2\right)\left(j^2-3kj+k^2\right)\\
&\times\left(-3j^2k+k^3+j^3\right)\left(j^4-4kj^3+5j^2k^2-k^4\right)\\
&\times\left(-2kj^3+j^4-j^2k^2+k^4\right)\left(j^4-4kj^3+j^2k^2+2k^3j-k^4\right)\\
&\times\left(j^2-kj+k^2\right)(j-k)^3,
\end{aligned}$$

which is independent of t. If we take $j := 2$ and $k := 3$, for example, then

$$\begin{aligned}
F_7 = &(t-7)(t-50)(t+24)(t+33)(t-13)(t+51)(t-38)\\
&-(t+7)(t+50)(t-24)(t-33)(t+13)(t-51)(t+38),
\end{aligned}$$

which expands to

$$F_7 = 13967553600.$$

Size 8. The following is a (homogeneous) size 8 solution due to Chernick [1937]:

$$\begin{aligned}
F_8 := &\left(t^2-R_1^2\right)\left(t^2-R_2^2\right)\left(t^2-R_3^2\right)\left(t^2-R_4^2\right)\\
&-\left(t^2-R_5^2\right)\left(t^2-R_6^2\right)\left(t^2-R_7^2\right)\left(t^2-R_8^2\right),
\end{aligned}$$

where

$$\begin{aligned}
R_1 &:= 5m^2+9mn+10n^2,\\
R_2 &:= m^2-13mn-6n^2,\\
R_3 &:= 7m^2-5mn-8n^2,\\
R_4 &:= 9m^2+7mn-4n^2,\\
R_5 &:= 9m^2+5mn+4n^2,\\
R_6 &:= m^2+15mn+8n^2,\\
R_7 &:= 5m^2-7mn-10n^2,\\
R_8 &:= 7m^2+5mn-6n^2.
\end{aligned}$$

On expansion,

$$\begin{aligned} F_8 = &- 10752mn(2n+m)(n+m)(2n+3m) \\ &\times (n+2m)(4n-m)(5n+4m)(n-2m)(3n+m) \\ &\times (n-m)(n+5m)\left(3n^2+2mn-2m^2\right)\left(n^2+mn+m^2\right). \end{aligned}$$

Size 9. We know no parametric solution of size 9. Indeed, only two inequivalent solutions are known. Both are symmetric, and they are the following:

$$\begin{aligned} &[-98,-82,-58,-34,13,16,69,75,99] \\ &\quad =_8 [98,82,58,34,-13,-16,-69,-75,-99] \end{aligned}$$

and

$$\begin{aligned} &[174,148,132,50,8,-63,-119,-161,-169] \\ &\quad =_8 [-174,-148,-132,-50,-8,63,119,161,169]. \end{aligned}$$

There are no other symmetric size 9 solutions of height less than 2000. (The height is the entry of largest modulus.)

Size 10. There are two small size 10 solutions known. They are

$$[\pm 99, \pm 100, \pm 188, \pm 301, \pm 313] =_9 [\pm 71, \pm 131, \pm 180, \pm 307, \pm 308]$$

and

$$[\pm 103, \pm 189, \pm 366, \pm 452, \pm 515] =_9 [\pm 18, \pm 245, \pm 331, \pm 471, \pm 508].$$

Otherwise, no symmetric examples of height less than 1500 exist.

The following size 10 example is originally due to Letac and is much simplified in Smyth [1991]. It constructs an infinite family of inequivalent ideal size 10 solutions based on rational solutions of an elliptic curve.

Let

$$\begin{aligned} F_{10} := &\left(t^2-R_1^2\right)\left(t^2-R_2^2\right)\left(t^2-R_3^2\right)\left(t^2-R_4^2\right)\left(t^2-R_5^2\right) \\ &-\left(t^2-R_6^2\right)\left(t^2-R_7^2\right)\left(t^2-R_8^2\right)\left(t^2-R_9^2\right)\left(t^2-R_{10}^2\right), \end{aligned}$$

where

$$\begin{aligned} R_1 &:= (4n+4m), & R_2 &:= (mn+n+m-11), \\ R_3 &:= (mn-n-m-11), & R_4 &:= (mn+3n-3m+11), \\ R_5 &:= (mn-3n+3m+11), & R_6 &:= (4n-4m), \\ R_7 &:= (-mn+n-m-11), & R_8 &:= (-mn-n+m-11), \\ R_9 &:= (-mn+3n+3m+11), & R_{10} &:= (-mn-3n-3m+11). \end{aligned}$$

On expansion of F_{10}, the constant coefficient is a polynomial in n and m alone. The rest of the expansion is divisible by the factor

$$m^2n^2 - 13n^2 + 121 - 13m^2.$$

Thus, any solution of the above biquadratic gives a size 10 solution. One such solution is given by $n = 153/61$ and $m = 191/79$. A second solution is given by $n = -296313/249661$ and $m = -1264969/424999$. The first of these gives the following solution:

$$[\pm 12, \pm 11881, \pm 20231, \pm 20885, \pm 23738]$$
$$=_9 [\pm 436, \pm 11857, \pm 20449, \pm 20667, \pm 23750].$$

The above biquadratic is equivalent to the elliptic curve

$$y^2 = (x - 435)(x - 426)(x + 861)$$

and gives rise to infinitely many inequivalent solutions. See Smyth [1991].

Size 11. No solutions are known, and no ideal symmetric solutions with all entries of modulus less than 2000 exist. See Borwein, Lisoněk, and Percival [to appear] and the last section of this chapter.

Size 12. The only known size 12 solution, found by Nuutti Kuosa and Chen Shuwen, is

$$[\pm 151, \pm 140, \pm 127, \pm 86, \pm 61, \pm 22] =_{11} [\pm 148, \pm 146, \pm 121, \pm 94, \pm 47, \pm 35].$$

There are no other symmetric solutions with all entries of modulus less than 1000.

Searching for Solutions

At present there are no known methods for finding ideal symmetric solutions of size 11 or higher to the Prouhet–Tarry–Escott problem other than massive searches. Nevertheless, the required searches can be made significantly less massive than the naive approach. (See Borwein, Lisoněk, and Percival [to appear].)

To begin with, ideal symmetric solutions of size $2n$ and $2n + 1$ are defined uniquely by $n + 1$ elements. In the case of a solution of even size, given $\alpha_1, \ldots, \alpha_{n+1-k}$ and $\beta_1, \ldots, \beta_k$, we note that as

$$\prod_{i=1}^{n}(z^2 - \alpha_i^2) - \prod_{i=1}^{n}(z^2 - \beta_i^2) = C,$$
$$\prod_{i=1}^{n}(\beta_j^2 - \alpha_i^2) - 0 = C \text{ for } j = 1, \ldots, n,$$

and so

$$\frac{1}{C}\prod_{i=n-k+2}^{n}(\beta_j^2-\alpha_i^2)=\prod_{i=1}^{n-k+1}(\beta_j^2-\alpha_i^2)^{-1} \text{ for } j=1,\dots,k,$$

which gives us k evaluations of the unique degree $k-1$ polynomial with leading coefficient $1/C$ and roots $\alpha_{n-k+2},\dots,\alpha_n$. These points can thus be interpolated, and the resulting polynomial solved to yield the unspecified α_i. The remaining β_i can be computed similarly. This reduces the dimension of the problem in the even case from $2n$ to $n+1$.

In an analogous manner, given $\alpha_1,\dots,\alpha_{n+1}$ of an ideal symmetric size $2n+1$ solution to the Prouhet–Tarry–Escott problem, we note that as

$$\prod_{i=1}^{2n+1}(z+\alpha_i)-\prod_{i=1}^{2n+1}(z-\alpha_i)=C,$$

$$\prod_{i=1}^{2n+1}(\alpha_j+\alpha_i)=C \text{ for } j=1,\dots,n+1,$$

and so

$$\frac{1}{C}\prod_{i=n+2}^{2n+1}(\alpha_j+\alpha_i)=\prod_{i=1}^{n+1}(\alpha_j+\alpha_i)^{-1} \text{ for } j=1,\dots,n+1,$$

which again uniquely specifies a polynomial that can be interpolated and solved to give the unknown α_i. This reduces the dimension of the problem in the odd case from $2n+1$ to $n+1$.

In addition to reducing the search space from $2n$ or $2n+1$ dimensions to $n+1$ dimensions, we can reduce the search space further by considering the modular properties of solutions. Each size of solution has associated with it a set of primes that must divide the constant C (see C1). For odd sizes, if a prime p divides C, then (subject to reordering of the α_i) we must have $\alpha_1\equiv 0 \pmod p$ and $\alpha_{2k}+\alpha_{2k+1}\equiv 0 \pmod p$, while for even sizes the equivalent requirement is that $\alpha_k^2\equiv\beta_k^2 \pmod p$.

The best known approach to finding ideal symmetric solutions to the PTE problem is thus to find all $(n+1)$-tuples satisfying the divisibility criteria (for the appropriate size), and test whether they extend to solutions of size $2n$ or $2n+1$.

The following searches were done using the method described above and approximately 10^{17} floating-point operations on 100 relatively fast PCs (by 2001 standards a large computation). The method lends itself to trivial parallelization with essentially no communication needed between processors.

Size	Search limit	Result
9	2000	one (inequivalent) solution found
10	1500	two (inequivalent) solutions found
11	2000	no solutions found
12	1000	one (inequivalent) solutions found

Introductory Exercises

E1. Prove Theorem 1.

E2. Show that if $[\alpha_1, \ldots, \alpha_n]$ and $[\beta_1, \ldots, \beta_n]$ is an ideal solution and is ordered such that $\alpha_1 \le \alpha_2 \le \cdots \le \alpha_n$ and $\beta_1 \le \beta_2 \le \cdots \le \beta_n$, then $\alpha_i \ne \beta_j$ for any i and j and

$$\alpha_1 < \beta_1 \le \beta_2 < \alpha_2 \le \alpha_3 < \beta_3 \le \beta_4 < \alpha_4 \cdots$$

(where without loss of generality we assume that $\alpha_1 < \beta_1$).

Conclude that an ideal solution of the Prouhet–Tarry–Escott problem (in the third equivalent form) is a polynomial of height at most 2. Conclude also that $k = n - 1$ is best possible in the first theorem of this chapter.

E3. Show that for each prime p, the Prouhet–Tarry–Escott problem of size p has nontrivial solutions mod p.

E4. Show that the parametric solutions of this section give rise to infinitely many inequivalent solutions.

E5. There are various results concerning the divisibility of

$$C_n := \prod_{i=1}^{n}(z - \alpha_i) - \prod_{i=1}^{n}(z - \beta_i),$$

where $[\alpha_i] =_{n-1} [\beta_i]$ is an ideal solution. Prove the following lemma.

Lemma. *If $[\alpha_i] =_{n-1} [\beta_i]$ is an ideal solution with C_n defined as above, then*

$$|C_n| = \left|\prod_{i=1}^{n}(\beta_j - \alpha_i)\right| = \left|\prod_{i=1}^{n}(\alpha_j - \beta_i)\right| = \left|\prod_{i=1}^{n}\alpha_i - \prod_{i=1}^{n}\beta_i\right| = \left|\frac{\sum_{i=1}^{n}\alpha_i^n - \sum_{i=1}^{n}\beta_i^n}{n}\right|$$

for all j.

E6. Suppose

$$f(z) := \sum_{i=1}^{n} z^{\alpha_i} - \sum_{i=1}^{n} z^{\beta_i}$$

is divisible by

$$\prod_{i=1}^{k}(1 - z^{n_i}).$$

Show that

$$k! \prod_{i=1}^{k} n_i \;\Big|\; \sum_{i=1}^{n}\alpha_i^k - \sum_{i=1}^{n}\beta_i^k.$$

Computational Problems

C1. For fixed p, find ideal solutions of the Prouhet–Tarry–Escott problem mod p. Show that the constant C in the second equivalent form of the problem is divisible by a set of primes that depends only on p. For example, for $p = 11$, the constant C is divisible by 2, 3, 5, 7, 11, 13, and 17. For $p = 13$, the constant C would have to be divisible by all primes up to 31. (See Borwein, Lisoněk, and Percival [to appear] and Rees and Smyth [1990].)

C2. Find all symmetric solutions of sizes 1 through 5 in parametric form.

C3. Verify that it is possible to solve the even symmetric problem of size 6 in Maple (or equivalent). The following simple Maple code finds a parametric solution to the even symmetric problem of size 6. (Actually, it is a translated solution with $a_6 = 0$.)

```
PTE:=proc(n)
   local i,j,k,S;
   S:={seq(a[j]=a[1]-a[n+1-j],j=n/2+1..n),
       seq(b[j]=a[1]-b[n+1-j],j=n/2+1..n)};
subs(S,{seq(sum(a[i]^k,i=1..n)-sum(b[i]^k,i=1..n),k=1..n-1)});
end;
```

The command `solve(PTE(6))` gives the following as rational solutions of size 6:

$$\Big\{a_2 = a_2, b_1 = b_1, b_3 = b_3, b_2 = \frac{a_2^2 - a_2 b_3 + b_1 b_3 - a_2 b_1}{-b_1 - b_3 + a_2},$$
$$a_1 = \frac{2}{3}\frac{a_2^2 - b_3^2 - b_1^2 - b_1 b_3}{-b_1 - b_3 + a_2}, a_3 = \frac{-b_1^2 - b_1 b_3 + a_2 b_1 + a_2 b_3 - b_3^2}{-b_1 - b_3 + a_2}\Big\}.$$

Show that the three-parameter solution of size 6 of this chapter (in nonsymmetric form) is just a reworking of the above output.

Research Problems

R1. Find infinite families of ideal solutions of the Prouhet–Tarry–Escott problem of size 9 and size 12 or show they can't exist.

R2. Find an ideal solution of size 11 or any size greater than 12.

R3. Show for some n that no ideal solutions of the Prouhet–Tarry–Escott problem exist.

Selected References

1. P. Borwein and C. Ingalls, *The Prouhet–Tarry–Escott problem revisited*, Enseign. Math. (2) **40** (1994), 3–27.

2. P. Borwein, P. Lisoněk and C. Percival, *Computational investigations of the Prouhet–Tarry–Escott problem* (to appear).

3. W.H.J. Fuchs and E.M. Wright, *The 'easier' Waring problem*, Quart. J. Math. Oxford Ser. **10** (1939), 190–209.

4. E. Rees and C.J. Smyth, *On the constant in the Tarry–Escott problem*, in *Cinquante Ans de Polynômes*, Springer-Verlag, Berlin, 1990.

5. E.M. Wright, *Prouhet's 1851 solution of the Tarry–Escott problem of 1910*, Amer. Math. Monthly **66** (1959), 199–201.