18.317 Combinatorics, Probability, and Computations on Groups November 2, 2001

Lecture 21

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1 Dirichlet forms and mixing time

Let G be a finite group, and let V be the vector space of real-valued functions from G. There is a natural inner product on this space

$$\langle \phi, \varphi \rangle = \sum_{x \in G} \varphi(x)\psi(x) = |G| \mathbf{E} \varphi(X)\psi(X)$$

where X is chosen from g according to uniform measure. Let π be a probability distribution on G, and let P_π denote the transition kernel of the random walk $\{X_n\}$ that moves from x to xy in every step, where y is distributed according to π. Just like any other transition kernel, P_π acts on the space of functions on G as follows

$$[P_\pi \varphi](x) = \sum_{y \in G} \varphi(xy)\pi(y) = \mathbf{E}[\varphi(X_1) \mid X_0 = x].$$

Define the support of $\varphi \in V$ in the usual way,

$$\mathrm{supp}(\varphi) = \{x \in G \mid \varphi(x) \neq 0\}.$$

We now define the Dirichlet form

$$\mathcal{E}_\pi(\varphi, \varphi) = \langle (I - P_\pi)\varphi, \varphi \rangle = |G| \mathbf{E}\left[(\varphi(X_0) - \varphi(X_1))\, \varphi(X_0) \right]$$

where now X_0 is chosen according to uniform distribution on G. Note that if Z_0, Z_1 are real-valued random variables with the same distribution, then

$$\mathbf{E}[(Z_0 - Z_1)Z_0] = \frac{1}{2}\mathbf{E}(Z_0 - Z_1)^2.$$

Since the uniform distribution is stationary with respect to convolution, X_0 and X_1 have the same distribution, and we may apply this to $\varphi(X_0)$, $\varphi(X_1)$ to get the alternative formula for the Dirichlet form

$$\mathcal{E}_\pi(\varphi, \varphi) = \frac{|G|}{2}\mathbf{E}(\varphi(X_0) - \varphi(X_1))^2 = \frac{1}{2}\sum_{x,y \in G} (\varphi(x) - \varphi(xy))^2 \pi(y).$$

Now let $\Gamma = \Gamma(G, S)$ be a Cayley graph of G with respect to a symmetric generator set S. Let γ be a function that assigns to each $y \in G$ a path from the identity to y in Γ. We will assume that γ is geodesic, that is its values are shortest paths. Let $\mu_s(y) = \mu_s(y, \gamma)$ denote the number of times a generator s appears in the decomposition

$$y = s_1 s_2 \ldots s_\ell \tag{1}$$

along the path $\gamma(y)$. For $C \subseteq G$ Let

$$N_\gamma(s, C) = \max_{x,y \in C} \mu_s(x^{-1}y).$$

We have a following version of a theorem by Diaconis and Saloff-Coste.

Theorem 1 (Comparison of Dirichlet forms) *Let $C \subseteq G$, let $\overline{C} = C \cup \partial C$, and let $d = \mathrm{diam}(\overline{C})$. Consider π, $\tilde{\pi}$ symmetric probability distributions on G, and let $S \subseteq \mathrm{supp}(\pi)$. Then*

$$\mathcal{E}_\pi(\varphi, \varphi) \geq \frac{1}{A}\mathcal{E}_{\tilde{\pi}}(\varphi, \varphi)$$

where

$$A = d \max_{s \in S} \frac{N_\gamma(s, \overline{C})}{\pi(s)}.$$

Proof: Let $y \in G$, and write y in the form (1). We can write

$$\varphi(x) - \varphi(xy) = [\varphi(xs) - \varphi(xs_1)] + \ldots + [\varphi(xs_1 \ldots s_{\ell-1} - \varphi(xy)].$$

It follows, for example, by the Cauchy-Schwarz inequality that

$$(\varphi(x) - \varphi(xy))^2 \leq \ell^* \sum_{i=1}^{\ell} (\varphi(xs_1 \ldots s_{i-1}) - \varphi(xs_1 \ldots s_i))^2$$

where ℓ^* is the number of nonzero terms in the sum, and is bounded above by d =diam$(\overline{C})$, since γ is geodesic. Summing this inequality over $x \in G$ we get

$$\sum_{x \in G} (\varphi(x) - \varphi(xy))^2 \leq d \sum_{z \in G, s \in S} N_\gamma(s, \overline{C})(\varphi(z) - \varphi(zs))^2.$$

Since this holds for all $y \in G$, we may average the left hand side with respect to y with weights $\tilde{\pi}(y)$ to get

$$\sum_{x,y \in G} (\varphi(x) - \varphi(xy))^2 \tilde{\pi}(y) \leq d \sum_{z \in G, s \in S} N_\gamma(s, \overline{C})(\varphi(z) - \varphi(zs))^2$$

Finally, the right hand side is clearly bounded by

$$A \sum_{z \in G, s \in S} (\varphi(z) - \varphi(zs))^2 \pi(s)$$

and the statement of the theorem follows. ■

By the way, the symmetry assumption for S was not used.