HOMEWORK 1 (18.319, FALL 2006)

1) Let $X_d \subset \mathbb{R}^d$ be a union of $(2d+1)$ unit cubes, where one cube in the center is attached to all others by a $(d-1)$-dimensional face.

a) Check that X_2 can tile $\mathbb{R}^2$.

b) Prove or disprove that X_3 can tile $\mathbb{R}^3$.

c) Find all $d \geq 4$ for which X_d can tile $\mathbb{R}^d$.

2) Let T_1, T_2 be two triangulations of a polygon $P \subset \mathbb{R}^2$. A *move* $T_1 \longleftrightarrow T_2$ is a transformation changing a triangle into a union of two (see Figure 1 below).

a) Prove that every triangulation of a triangle can be connected to an empty triangulation by a sequence of moves.

b) Prove that every two triangulations of a convex polygon are connected by a sequence of moves.

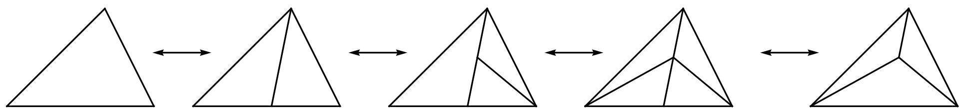

FIGURE 1. A sequence of moves on triangulations.

3) A convex set $X \subset \mathbb{R}^2$ is called *round* if the length of its projection on every line is equal to 1. For example, a circle with diameter 1 is round.

a) Find an infinite family of examples of round convex sets.

b) Prove that every round convex set has perimeter π.

4) Let $P \subset \mathbb{R}^3$ be a convex polytope and let $z \in P$ be an interior point. Prove that there exist a face F of P such that the orthogonal projection of z onto F lies in the interior of F. Find a non-convex polytope for which this is false.

5) Let $X \subset \mathbb{R}^2$ be a convex set on a plane with a smooth boundary. We say that a line cutting X is *special* if it is orthogonal to the boundary at two points of intersection. Prove that there exists at least two special cuts. (Of course, there are infinitely many special cuts of a circle.)

6) Prove the Borsuk conjecture for centrally-symmetric convex sets in $\mathbb{R}^3$. Same for $\mathbb{R}^d$.

7) Prove that a plane cannot be tiles with a copies of the same convex octagon. Show that is possible when the octagon is non-convex.

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This Homework is due Wednesday September 27 at 13:05 pm. Please make sure you have read the collaboration policy on the course web page.

P.S. Do not forget: some of these problems are quite difficult. By no means you are expected to solve all or even most of them.