HOMEWORK 2 (18.314, FALL 2006)

Definition. Let $[n] = \{1, 2, \ldots, n\}$, and let $\begin{bmatrix} n \\ k \end{bmatrix}$ be a set of all k-subsets. For every $A \subset [n]$, define $\mathrm{inv}(A) = |\{(i,j) : 1 \le i < j \le n,\, i \notin A,\, j \in A\}|$. Let

$$\binom{n}{k}_q = \sum_{A \in \begin{bmatrix} n \\ k \end{bmatrix}} q^{\mathrm{inv}(A)}$$

be the *q-binomial coefficients.*

Definition. Let $(i)_q = (q^i - 1)/(q - 1)$, and $(n!)_q = (1)_q (2)_q \cdots (n)_q$.

1) Draw the first 6 lines of the q-Pascal triangle containing polynomials $\binom{n}{k}_q$. Find a recurrence relation for these polynomials. Prove by induction that

$$\frac{(n!)_q}{(k!)_q \cdot (n-k)!_q}$$

satisfy these recurrence relations.

2) Prove by induction that $\binom{n}{k}_q$ satisfy the same recurrence relations. Conclude that

$$\binom{n}{k}_q = \frac{(n!)_q}{(k!)_q \cdot (n-k)!_q}$$

3) Recall the bijection between $\begin{bmatrix} n \\ k \end{bmatrix}$ and grid paths. Show that the number of inversions in a k-subset of $[n]$ corresponds to the area under the grid path. Find another proof of 2).

4) Compute the number of permutations in S_n with exactly 2 inversions.

5) a) Compute the expected number of inversions in a permutation $\sigma \in S_n$.
b) Compute the expected number of inversions in k-subsets $A \in \begin{bmatrix} n \\ k \end{bmatrix}$.

6) Let A be a random k-subset of $[1024]$. Denote by p_k the probability that A does not contain any power of 2. Find a formula for p_k. Find the smallest k such that $p_k < 3/4$.

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This Homework is due Wednesday Sep 27 at 14:05 am.

Remember the collaboration policy: groups of at most four, write names on the solutions, only discussions are allowed, no copying.