A Theorem on Graphs

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A THEOREM ON GRAPHS.[1]

By Hassler Whitney.

I. Results of this paper.

1. Let a finite number of curves, or edges, whose end-points we call vertices, intersect at no other points than these vertices. Let the system be connected, that is, any two vertices are joined by a succession of edges, each two successive edges having a vertex in common. This forms a graph. A graph is planar if it can be mapped in a 1–1 continuous manner on a plane (or a sphere). If the vertices a, b are joined by an edge, we shall call the edge joining them ab, and shall say a touches b for short. A set of distinct vertices, $a, b, c, \cdots, e, f$, together with a set of distinct edges joining them in cyclic order, $ab, bc, \cdots, ef, fa$, we shall call a circuit.

A planar graph lying on the surface of a sphere divides this surface into a number of simply connected regions. The boundary of each of these regions may be a circuit. If so, we shall call these circuits elementary polygons. If all these polygons are n-gons, n fixed, we say the graph is composed of elementary n-gons.

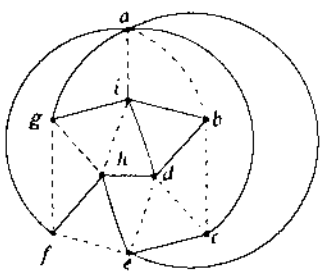

Fig. 1.

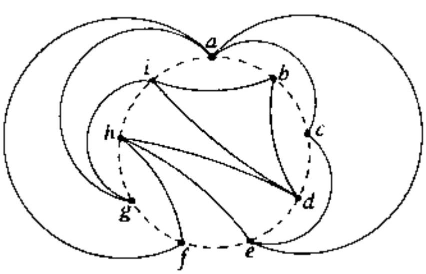

Fig. 2.

2. The fundamental theorem of this paper is the following:

Theorem I. *Given a planar graph composed of elementary triangles, in which there are no circuits of 1, 2, or 3 edges other than these elementary triangles, there exists a circuit which passes through every vertex of the graph.*

The problem of finding graphs for which this is so has been studied by several people.[2] This seems to be the first case when a large class of planar graphs has been shown to have this property.

3. This theorem gives immediately the following:

Normal Form. *Given any graph as described in Theorem* I, *containing*

[1] Received April 7, and July 14, 1930.—Presented to the American Mathematical Society, Febr. 22, 1930.

[2] See St. Laguë, A., Les Réseaux, Mémorial des Sciences Math., fasc. 18, Paris (1926).

n vertices, we can construct a graph homeomorphic with it as follows: Draw a regular polygon of n sides, and draw diagonals, no two of which cross, dividing the inside of the polygon into triangles. Similarly draw circular arcs, no two of which cross, dividing the outside of the polygon into circular triangles.

We have merely to find the circuit given by Theorem I, and distort it into the polygon.

4. A theorem on maps deducible immediately from Theorem I is the following, as we shall see later:

THEOREM II. *Given a map on the surface of a sphere containing at least three regions in which:*

(A_1) *The boundary of each region is a single closed curve without multiple point,*

(B) *Exactly three boundary lines meet at each vertex,*

(A_2) *No pair of regions taken together with any boundary lines separating them form a multiply connected region,*

(A_3) *No three regions taken together with any boundary lines separating them form a multiply connected region,*

we may draw a closed curve which passes through each region of the map once and only once, and touches no vertex.

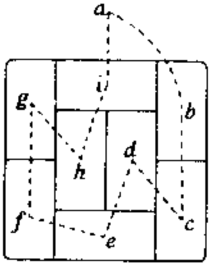

Fig. 3.

5. By means of Theorem II and a lemma to be proved, we have a solution of a conundrum, which we leave to the end of the paper.

6. Finally, Theorem I gives us a new statement of the four color map problem. Given any map on the surface of a sphere, we "color" it by assigning to each region a color in such a way that no two regions with a common boundary are of the same color. Given any polygonal configuration as described in 3., we "color" it by assigning to each vertex of the polygon a color in such a way that no two vertices which are joined by a line, either a side of the polygon or a diagonal, are of the same color.

EQUIVALENT STATEMENT OF THE FOUR COLOR MAP PROBLEM. *If every polygonal configuration as described in 3. can be colored in four colors, then every map on the surface of a sphere can be colored in four colors, and conversely.*

2. Proof of Theorem I.

We consider only the graphs defined in § 1, 2. As the graph is composed of elementary triangles, there are at least three vertices present. If there are only three, the theorem is obvious. We shall assume from here on that there are at least four vertices present.

As there are no circuits of one or two edges, no vertex touches itself, and any two vertices are joined by at most a single edge.

There is no vertex touching but a single other vertex. For then the boundary of the region surrounding this other vertex would not be a circuit, and therefor not an elementary triangle.

There is no vertex touching only two others. For suppose a touched b and c alone. Then the edges ab and ac would each be sides of two triangles, whose third sides are both edges bc. But there is only one edge bc, as two vertices are joined by at most a single edge. The two triangles thus cover the whole surface of the sphere, and there are thus only three vertices in the graph, contrary to hypothesis.

Consider a vertex a touching other vertices $b, c, \cdots, f$. We read the edges emenating from a in a counter-clockwise sense, and say, a touches $b, c, \cdots, f$ in cyclic order; or, a touches b, next c, $\cdots$, next f, next b. Remembering now that the graph is composed of elementary triangles, we have the three properties:

(α) *Each vertex touches at least three other vertices in cyclic order, distinct from each other and distinct from the first,*

(β) *If a touches b and next c, then b touches c and next a,*

(γ) *There are no triangles other than elementary triangles.*

These properties, together with the fact that the graph lies on a sphere, is all we need to prove the following lemma, from which the theorem is deduced.

LEMMA. *Consider a circuit R in a graph of the type considered in Theorem* I, *together with the vertices and edges on one side, which we shall call the inside. Let A and B be two distinct vertices of R, dividing R into the two parts R_1 and R_2, in each of which we include both A and B. Suppose*

(1) *No pair of vertices of R_1 touch each other inside R (are joined by an edge which lies inside R), and*

(2) *Either no pair of vertices of R_2 touch each other inside R, or else there is a vertex C in R_2 distinct from A and B, dividing R_2 into the two parts R_3 and R_4, in each of which we include C, such that no pair of vertices of R_3 and no pair of vertices of R_4 touch each other inside R.*

Then we can draw a line from A to B, passing only along edges of and inside R, and passing through each vertex of and inside R once and only once.

In brief, if we can divide the circuit R into either two or three parts, such that in any part, including end vertices, no pair of vertices touch each other inside R, we can then draw the required curve from any one end vertex to any other end vertex of these parts.

The theorem is an immediate consequence of the lemma. For consider any elementary triangle of the graph, containing the vertices A, B, C, which we call the circuit R. The rest of the graph we call the inside of the circuit. As each pair of vertices of R touch as a part of the circuit, and any two vertices are joined by at most one edge, it follows that no pair of them touch inside R. Thus the conditions of the lemma are fulfilled, and we can pass from A to B through every vertex of R and every vertex inside R, that is, through every vertex of the graph. We now pass from B directly to A, forming a closed curve. The edges passed over by the curve form the desired circuit.

Proof of the lemma. Assume the lemma is true for all circuits which, with the vertices inside, contain m vertices, $m = 3, 4, \cdots, n-1$. It is obviously true for the case where $m = 3$. We will prove it for all circuits which, with the vertices inside, contain n vertices. Then, by mathematical induction, it is true in general.

Take any circuit R therefore, which, with the vertices inside, contains n vertices. Let the vertices of the circuit be $A, a_1, a_2, \cdots, a_\alpha, B, b_1, b_2, \cdots, b_\beta, C, c_1, c_2, \cdots, c_\gamma, A$, (reading in a clockwise sense). We assume that no pair of the vertices $A, a_1, \cdots, a_\alpha, B$, no pair of the vertices $B, b_1, \cdots, b_\beta, C$, (or with C replaced by A, if there is no C), and no pair of the vertices $C, c_1, \cdots, c_\gamma, A$ touch inside the circuit. The vertices $C, c_1, \cdots, c_\gamma$ may be missing from the circuit, as may also the vertices $a_1, \cdots, a_\alpha$ or $b_1, \cdots, b_\beta$. We wish to draw the required curve from A to B.

We will divide the proof into four parts, according to what pairs of vertices of the circuit touch inside the circuit:

Case (1). Some vertex a_q touches a vertex b_r, C, or c_s inside R.

Case (2). There are no edges of the above form, but either B touches a vertex c_s or A touches a vertex b_r inside R.

Case (3). No pairs of vertices of the circuit touch inside the circuit.

Case (4). Some vertex b_r touches a vertex c_s inside R, but there are no edges of other forms between vertices of the circuit inside the circuit.

Case (1). Assume there is an edge of one of the forms $a_q C$, $a_q c_s$. The case where there is no edge as above, but there is an edge of the form $a_q b_r$, is reduced to this case by interchanging the rôles of A and B and of c_s and b_r. Suppose the edge nearest A is $a_i c_k$. If it is $a_i C$, we call C, c_k. The meaning of "nearest A" is obvious. Now either, Case (1a), c_k touches none of the vertices $a_{i+1}, \cdots, a_\alpha, B$, or, Case (1b), a_i touches none of the vertices $C, c_1, \cdots, c_{k-1}$ inside the circuit $c_k, a_i, \cdots, a_\alpha, B, b_1, \cdots, b_\beta, C, c_1, \cdots, c_k$. If c_k is C, the latter condition is satisfied automatically.

Consider Case (1a). We shall draw the required curve in two steps: first from A to c_k, then from c_k to B.

If first, Case (1a$_1$), a_i is not a_1, a_{i-1} exists, and does not touch c_k inside the circuit, as the edge $a_i c_k$ was the edge of this form nearest A. Therefore a_i must touch some vertex in between a_{i-1} and c_k. For if a_i touched a_{i-1} and next c_k, a_{i-1} would touch c_k and next a_i, by (β). Thus a_{i-1} would touch c_k between a_{i-2} (or A) and a_i, and the edge $a_{i-1}\,c_k$ would therefor be inside the circuit, which it cannot be, again as the edge $a_i c_k$ was the edge of this form nearest A. As a_i touches no vertices of the set $c_{k+1}, \cdots, c_\gamma$, $A, a_1, \cdots, a_{i-1}$ inside the circuit, any vertices it touches between a_{i-1} and c_k must be vertices inside the circuit R. Call them in order $p_1, p_2, \cdots, p_\varphi$. Then, by (β), a_{i-1} touches p_1, p_1 touches $p_2, \cdots$, and p_φ touches c_k. We have

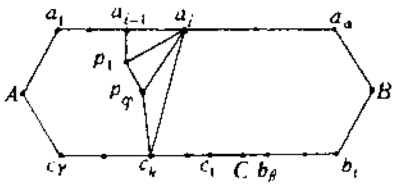

Fig. 4.

thus formed a circuit $A, a_1, \cdots, a_{i-1}, p_1, \cdots, p_\varphi, c_k, \cdots, c_\gamma, A$. No pair of the vertices $A, a_1, \cdots, a_{i-1}$ touch inside this circuit, as none of the set $A, a_1, \cdots, a_\alpha, B$ touched inside the circuit R. Similarly no pair of the set $c_k, \cdots, c_\gamma, A$ touch inside the circuit. Finally, no pair of the set $a_{i-1}, p_1, \cdots, p_\varphi, c_k$ touch inside the circuit. For suppose for instance p_g touched p_h inside, $h > g$. a_i does not touch p_g and next p_h, as p_g and p_h would then touch as a part of the circuit, and therefor not inside the circuit. Therefor a touches a vertex p_s in between. But then a_i, p_g and p_h form a triangle, with p_s on one side, and other vertices, as A, on the other side, which is therefor not an elementary triangle, in contradiction to (γ). Thus all the conditions of the lemma are satisfied for this circuit, and there are fewer than n vertices in and within the circuit. We can therefor draw a line from A to c_k passing through every vertex of and inside the circuit.

If next a_i is a_1, suppose, Case ($1a_2$), c_k is not c_γ. (If the edge nearest A is $a_1\,C$, suppose there is a vertex c_l in R.) By hypothesis, c_k does not touch A inside the circuit. Therefor a_1 touches vertices between A and c_k. For otherwise, a_1 would touch A and next c_k, and therefor A would touch c_k and next a_1, by (β). But as c_k is not c_γ, A would touch c_k between c_γ and a_1, and the edge Ac_k would be inside the circuit. As a_1 does not touch $c_{k+1}, \cdots, c_\gamma$ inside the circuit, the vertices it touches between A and c_k must be vertices not in R. Call these vertices in order $p_1, \cdots, p_\varphi$. We get thus a circuit $A, p_1, \cdots, p_\varphi, c_k, \cdots, c_\gamma, A$. No pair of the set of vertices $c_k, \cdots, c_\gamma, A$ touch inside the circuit, nor do any of the set A, $p_1, \cdots, p_\varphi, c_k$, using exactly the same reasoning as in Case ($1a_1$). Thus the lemma applies to this circuit, and we pass from A to c_k, passing through every vertex of and inside the circuit.

In each of the two Cases ($1a_1$) and ($1a_2$) we have now passed through every vertex of and inside R which is on A's side of the edge $a_i c_k$. For consider the circuit $a_i, c_k, p_\varphi, \cdots, p_1, a_{i-1}, a_i$, (or with a_{i-1} replaced by A, if a_i is a_1). As a_i touches every other vertex of the circuit, there can be no vertices inside the circuit. For if there were a vertex d inside the circuit, it must then lie inside one of the triangles a_i, p_1, a_{i-1} (or A), a_i, or a_i, p_2, p_1, a_i, or $\cdots$, or a_i, c_k, p_φ, a_i. In any case, (γ) would be violated. We have thus only to pass from c_k to B on B's side of the edge $a_i c_k$, that is, through the circuit $c_k, a_i, \cdots, a_\alpha, B, b_1, \cdots, b_\beta, C, c_1, \cdots, c_k$.

We have still to consider in Case (1a) the Case ($1a_3$), where the edge nearest A was the edge $a_1 c_\gamma$ (or $a_1 C$, when there is no c_γ). Draw a line directly from A to c_γ (or C). As there are no vertices inside the circuit A, a_1, c_γ, A (or A, a_1, C, A) by (γ), we have left to pass through only vertices of and inside the same circuit as in Cases ($1a_1$) and ($1a_2$).

But we can do this, by the lemma. For, no pair of the set $a_i, \cdots, a_\alpha, B$ touch inside the circuit. Also, c_k touches none of these vertices inside the circuit, by the hypothesis of Case (1a). Therefor none of the vertices $c_k, a_1, \cdots, a_\alpha, B$ touch inside the circuit. Nor do any of the set $B, b_1, \cdots, b_\beta, C$, or any of the set $C, c_1, \cdots, c_k$, (if these are present), by the original hypotheses. The circuit is thus divided into two or three parts, depending on whether c_k is C or not, and the lemma applies in either case. We thus pass from c_k to B, completing the required curve from A to B. This disposes of Case (1a).

Consider Case (1b), where a_i touches none of the vertices $C, c_1, \cdots, c_{k-1}$, inside the circuit (if any are present). In this case, instead of passing from A to c_k through all the vertices of and inside the circuit $A, a_1, \cdots, a_i, c_k, \cdots, c_\gamma, A$, except a_1, the same steps show we can pass from A to a_i through every vertex of and inside this circuit except c_k. We now apply the lemma to pass from a_i to B. For, no pair of vertices of the set $C, c_1, \cdots, c_k$ touch inside the circuit $a_i, \cdots, a_\alpha, B, b_1, \cdots, b_\beta, C, c_1, \cdots, c_k, a_i$, and a_i touches none of these vertices inside the circuit; therefor none of the set $C, c_1, \cdots, c_k, a_i$ touch inside the circuit. Also, no pair of the set $a_i, \cdots, a_\alpha, B$, and no pair of the set $B, b_1, \cdots, b_\beta, C$ touch inside the circuit. The proof for Case (1) is now complete.

Case (2). Suppose B touches a vertex c_s inside the circuit. Of all such vertices, let the one nearest A be c_k. Exactly as we before passed from A to c_k, going through all the vertices on A's side of the edge $a_i c_k$, we now pass from A to c_k, going through all the vertices on A's side of the edge Bc_k. We have now only to pass from c_k to B, going through all the vertices on the other side of the edge Bc_k. But we can do this, by the lemma. For the vertices c_k, B do not touch inside the circuit $c_k, B, b_1, \cdots, b_\beta, C, c_1, \cdots, c_k$. Also, no vertices of the set $B, b_1, \cdots, b_\beta, C$, and no vertices of the set $C, c_1, \cdots, c_k$ touch inside the circuit.

The proof is the same if A touches some vertex b_r inside the circuit.

Case (3). No vertices of the circuit touch inside the circuit. As any circuit contains at least three vertices, there is at least one other vertex besides A and B in the circuit. Thus if we call the vertices of the circuit $A, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_\beta, A$, either a_1 or b_β, say b_β, is present. Draw a line from A to b_β. We have still to pass from b_β to B.

Suppose, Case (3a), a_1 is also present in the circuit. As a_1 and b_β do not touch inside the circuit, A does not touch b_β and next a_1, and A touches therefor other vertices in between. Calling these in order $p_1, \cdots, p_\varphi$, we have a circuit $b_\beta, p_1, \cdots, p_\varphi, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_\beta$, where at least b_β, p_1, a_1 and B are present. The lemma applies to this circuit. For, no pair of the vertices $b_\beta, p_1, \cdots, p_\varphi, a_1$, no pair of the vertices $a_1, \cdots, a_\alpha, B$, and no pair of the vertices $B, b_1, \cdots, b_\beta$ touch inside the circuit. There are no vertices inside the circuit $b_\beta, A, a_1, p_\varphi, \cdots, p_1, b_\beta$, as A touches all the other vertices of this circuit.

Suppose now, Case (3b), a_1 is not present in the circuit, but $b_1 \neq b_\beta$ is. Then, as B does not touch b_β inside the circuit, A touches vertices between b_β and B, and we obtain the circuit $b_\beta, p_1, \cdots, p_\varphi, B, b_1, \cdots, b_\beta$, to which the lemma applies. For, no pair of the vertices $b_\beta, p_1, \cdots, p_\varphi, B$, and no pair of the vertices $B, b_1, \cdots, b_\beta$ touch inside the circuit.

Consider now Case (3c), where the circuit R consists only of the vertices $A, B, b_1 = b_\beta, A$. If there are no vertices inside the circuit, we pass directly from b_β to B. If there are vertices inside the circuit, A touches vertices between b_β and B, and we obtain the circuit $b_\beta, p_1, \cdots, p_\varphi, B, b_\beta$, to which the lemma applies, as in Case (3b).

Case (4). No pair of vertices of the circuit R touch inside except for edges of the form $b_r c_s$. Of all such edges, let the one furthest from the vertex C be the edge $b_j c_k$. We will carry through the proof for this case in three steps:

(1) A chain of vertices $p_1, \cdots, p_\varphi$ with the edges joining them stretching from b_j to A and to a_1 (or to B, if there is no a_1), will be found.

(2) A subset of these vertices with the edges joining them will form another chain, $q_1, \cdots, q_\theta$.

(3) The required curve will be drawn from A to b_j on A's side of this latter chain, and from b_j to B on the other side of the chain.

(1) *The chain of p's.* As b_{j-1} (or B, if there is no b_{j-1}) does not touch c_k, the edge $b_j c_k$ being the one furthest from C, b_j touches a vertex in between, which is inside the circuit R. Call p_1 the vertex b_j touches just before c_k. Then p_1 touches c_k and forms the first vertex of the chain. If p_1 touches A, the first part of the chain is finished. If not, let c_{p_1} be the vertex of the set $c_k, \cdots, c_\gamma$ nearest A which it touches.

Suppose we have constructed the chain as far as the vertex p_i, which does not touch A, and c_{p_i} is the vertex nearest A which p_i touches. Assume the following properties hold:

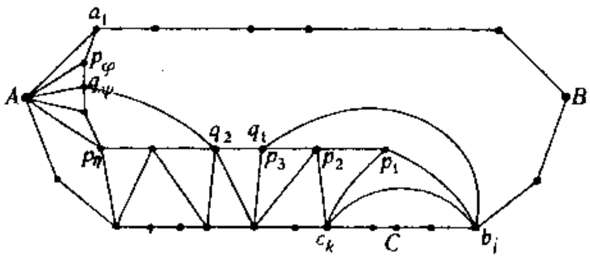

Fig. 5.

(a) All the p's are distinct.

(b) Each p_s, $s < i$, touches the vertex p_{s+1}, and each touches a vertex c_{p_s}.

(c) No p_s touches any of the vertices $c_{p_i}, \cdots, c_\gamma$ inside the circuit $A, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_j, p_1, \cdots, p_i, c_{p_i}, \cdots, c_\gamma, A$.

These properties are seen to hold when we have found the first vertex of the chain, p_1. Having found p_i, we find the next vertex, p_{i+1}, as follows. As p_i does not touch c_{p_i+1}, (or A, if c_{p_i} is c_γ), inside the circuit, c_{p_i} touches a vertex in between. Any such vertex is not a vertex of the circuit R, nor is it any of the vertices $p_1, \cdots, p_i$, by the above assumptions. Call p_{i+1} the vertex c_{p_i} touches next after p_i. If p_{i+1} touches A, the first part of the chain is finished. Otherwise, let $c_{p_{i+1}}$ be the vertex nearest A that p_{i+1} touches (which may be c_{p_i}). Now p_{i+1} is distinct from all former p's, p_i touches p_{i+1}, p_{i+1} touches $c_{p_{i+1}}$, and no vertex $p_1, \cdots, p_{i+1}$ touches $c_{p_{i+1}}$ or any vertex nearer A inside the new circuit. Thus the same properties still hold, and we continue finding vertices of the chain.

We note that, although p_{i+1} touches c_{p_i}, it touches no vertex c_s nearer C than c_{p_i}. Thus if p_i touches c_s, p_j touches c_t, and $j > i$, then $t \geqq s$.

We must eventually reach A. For each time a vertex p_i does not touch A, we find a new vertex p_{i+1}, all the vertices p_s are distinct, and there are only a finite number of vertices inside the circuit.

Call the last vertex of this chain p_η. If p_η touches a_1 (or B, if there is no a_1), call it also p_φ. Otherwise, A touches vertices in between, none of which are vertices of the circuit R or of the chain $p_1, \cdots, p_\eta$. Call these in order $p_{\eta+1}, \cdots, p_\varphi$. We now have a chain of vertices $p_1, \cdots, p_\varphi$, stretching from b_j to a_1(or B), each of which touches a vertex c_s or A.

(2) *The chain of q's.* Mark in now any edges there may be joining the vertices $b_j, p_1, \cdots, p_\varphi, a_1 (B)$ inside the circuit we now have, which includes the p's and B. Call q_1 the vertex of the set $p_1, \cdots, p_\varphi$ nearest $a_1 (B)$ which b_j touches (which may be p_1). Thus q_1 exists. Having found q_i, if it touches $a_1 (B)$, we cass call it q_θ. Otherwise, we take as q_{i+1} the vertex of the set $p_1, \cdots, p_\varphi$ nearest $a_1 (B)$ which q_i touches. Continue

in this manner till we reach $a_1(B)$. Now every vertex q_i touches a vertex c_s or A. Also, no vertices of the set $b_j, q_1, \cdots, q_\theta, a_1(B)$ touch inside the circuit $b_j, q_1, \cdots, q_\theta, a_1, \cdots, a_\alpha, B, b_1, \cdots, b_j$ (where the a's may be missing), on account of the construction of the chain. As, also, no pair of the vertices $a_1, \cdots, a_\alpha, B$, and no pair of the vertices $B, b_1, \cdots, b_j$ touch inside the circuit, we can apply the lemma and draw a line from b_j to B, passing through every vertex of and inside this circuit.

(3.) *The curve.* If there are no vertices q_s touching A, call $a_1(B)$, q_ψ. Otherwise, call the first vertex q_s which touches A, q_ψ. To finish the proof of the lemma, we have only to pass from A to b_j through every vertex on c_k's side of, but not in, the chain $b_j, q_1, \cdots, q_\psi, A$. For if q_ψ is $a_1(B)$, the chains $b_j, q_1, \cdots, q_\psi$ and $b_j, q_1, \cdots, q_\theta, a_1(B)$ are identical, and we have passed through every vertex of and on B's side of the chain in passing from b_j to B. If q_ψ is not $a_1(B)$, consider the circuit $A, a_1(B), q_\theta, \cdots, q_\psi, A$, (where q_ψ may be q_θ). As A touches each of these vertices, there can be no vertices inside the circuit, by (γ). Thus all the vertices we have not passed through on c_k's side of the chain $b_j, q_1, \cdots, q_\theta, a_1(B), A$, are also on c_k's side of the chain $b_j, q_1, \cdots, q_\psi, A$.

We will pass from A to b_j in two steps: first from A to c_k, on A's side of the edge $b_j c_k$, then from c_k to b_j, on C's side of the same edge.

Mark in all edges between the q's and the c's. Remembering that each vertex q_i, $i < \psi$, touches a vertex c_s, and that if q_i touches c_s, q_j touches c_t, and $j > i$, then $t \geqq s$, we see that these edges divide the section of the graph we must pass through into a number of sections, each of which we will pass through in turn.

Suppose q_ψ touches a vertex of the set $c_k, \cdots, c_\gamma$. Call the one nearest A that q_ψ touches c_g. If c_g is c_γ, there are no vertices inside the circuit A, q_ψ, c_γ, A, and we pass directly from A to c_γ. Otherwise, c_g does not touch A inside the circuit, and therefor q_ψ touches other vertices in between. Call these vertices in order $r_1, \cdots, r_\mu$. There are no vertices inside the circuit $A, q_\psi, c_g, r_\mu, \cdots, r_1, A$. Thus we need only pass from A to c_g through all the vertices of and inside the circuit $A, r_1, \cdots, r_\mu, c_g, \cdots, c_\gamma, A$. But we can do this, by the lemma. For, no pair of the vertices $A, r_1, \cdots, r_\mu, c_g$, and no pair of the vertices $c_g, \cdots, c_\gamma, A$ touch inside the circuit.

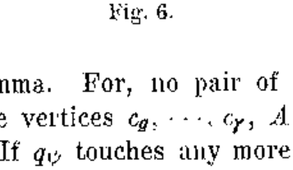

Fig. 6.

If q_ψ touches any more vertices of the set $c_k, \cdots, c_\gamma$, we pass through each of the sections thus formed in turn in exactly the same manner, till we reach the last c that q_ψ touches, c_h.

If the vertex nearest A of the c's that $q_{\psi-1}$ touches is c_i, we must now pass through the section bounded by c_h, q_ψ, $q_{\psi-1}$, $c_i, \cdots, c_h$.

If q_ψ did not touch any vertex c_s, we would have this section to pass through in the first place, c_h being replaced by A.

If c_i is c_h, this section is a triangle which contains no vertices inside, and we consider the next section. Suppose therefor c_i is not c_h. As then $q_{\psi-1}$ does not touch c_h, q_ψ touches vertices in between, none of which are any of the set $c_i, \cdots, c_h$. We obtain thus a chain of vertices stretching from c_h to $q_{\psi-1}$, of which the last is say d. Similarly, we obtain a chain of vertices stretching from q_ψ to c_i, of which the first is d. As there are no vertices inside the circuit c_h, q_ψ, $q_{\psi-1}$, $c_i, \cdots, d, \cdots, c_h$, we have only to pass from c_h to c_i through the circuit $c_h, \cdots, d, \cdots, c_i, \cdots, c_h$. We can do this, by the lemma. For, no vertices of the set $c_h, \cdots, d$, none of the set $d, \cdots, c_i$, and none of the set $c_i, \cdots, c_h$ touch inside the circuit.

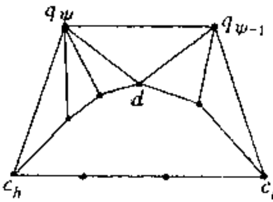

Fig. 7.

We pass in this manner through each section in turn, till we reach c_k. The last section, in particular, is bounded by the vertices c_f, q_1, b_j, $c_k, \cdots, c_f$, where c_f is either c_k or the vertex nearest c_k of the c's that q_1 touches. Thus here, b_j takes the place of what would otherwise be the next q.

We have now but to pass from c_k to b_j on C's side of the edge $b_j c_k$. We can do this, by the lemma. For, the vertices c_k, b_j, no pair of the set $b_j, \cdots, b_\beta$, C, and no pair of the set C, $c_1, \cdots, c_k$ touch inside the circuit thus described.

The proof of the lemma, and therefor of Theorem I, is now complete.

3. Proofs of the theorems on maps.

The dual representation. Given a map on the surface of a sphere, we find the dual representation in the form of a graph as follows. Mark in each region of the map a point, which will be a vertex of the graph, and which we shall call by the same name as the region of the map in which it lies. Across each boundary line of the map draw a line connecting the vertices in the two regions the boundary separates, forming an edge of the graph.

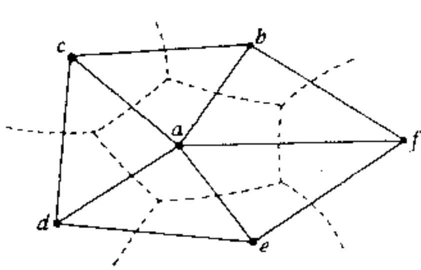

Fig. 8.

Now surrounding each vertex of the map there is a region of the graph bounded by a set of edges.

Proof of Theorem II. We will show first that in any map of the type considered in Theorem II, the dual graph holds to the properties (α), (β) and (γ) of § 2.

Each region of the map is simply connected, on account of (A_1). Each boundary is a boundary between two distinct regions. For suppose there were a boundary line QR running through a single region a. We could then, starting from a point P of QR, move into a on one side of QR, run along a path remaining always in a, and get back to P on the other side of QR. Let us now run around the boundary of a. At some time we pass along the boundary line QR. We are now inside the path we have drawn through a, and as the boundary of a is a closed curve, we must get out again. But we can only get out by passing through P, which contradicts (A_1).

Suppose we run around the boundary of a region a in a counter-clockwise sense. We are on successively sections of the boundary separating a from other regions $b, c, \cdots, f$, in cyclic order. Thus in the dual graph, a touches $b, c, \cdots, f$, in cyclic order, and these vertices are distinct from a.

Suppose a touches b and next c. Then if we pass around the boundary of the region a in a counter-clockwise sense, two successive sections of this boundary will be C, separating a and b, and B, separating a and c. C and B will meet at the vertex V. By (B), only one other boundary line abutts at V. Call it A. It must thus separate the regions b and c. Run now around the boundary of b in a counter-clockwise sense. Two successive sections of this boundary will be A and C. Thus we see that the vertex b touches c and next a, proving property (β).

Suppose now a touches in order $b, c, d, \cdots, f$. These vertices are then all distinct. For consider any two of the vertices a touches, say b and d. If a touches b and next d, or d and next b, then b touches d, and therefor b and d are distinct. Suppose now a touches a vertex c after b and before d, and a vertex f after d and before b. Here again b and d must be distinct, for otherwise the regions a and b would form a multiply connected region, separating c and f, contrary to (A_2).

Except in a map of three regions, for which Theorem II is obvious, each region of the map touches at least three others. For if there were a region touching only one or two others, that region or pair of regions would form a multiply connected region, contrary to (A_1) or (A_2). Thus each vertex of the dual graph touches at least three others. This finishes the proof of property (α).

Finally, there are no triangles in the graph other than elementary triangles. For if there were such a triangle, the regions of the map surrounding it would form a multiply connected region, contrary to (A_3). The properties (α), (β) and (γ) are now proved.

Now, applying Theorem I to the dual graph, we find a circuit passing through every vertex of the graph. This circuit is the desired closed curve passing through every region of the map.

Proof of the equivalent statement of the four color map problem. Elementary considerations in the four color map problem show that if any map of the type considered in Theorem II can be colored in four colors, then any map on the surface of a sphere can be colored in four colors. We need therefor consider only maps of the above type.

Put the dual graph of such a map in the normal form. Suppose we can color this polygonal configuration in four colors. We then color each region of the map with the same color as the corresponding vertex of the dual graph. Any two regions with a common boundary correspond to two vertices of the graph which are joined by an edge, and are therefor of different colors.

The converse is obvious, as every polygonal configuration is the dual of a map.

Conundrum. Suppose a man, living in a certain country (state), wishes to visit all the countries about him, but does not wish to pass through any country more than once on his voyage. Can he do it? If the region he wishes to visit covers the entire globe, he can do it if the countries make up a map of the type considered in Theorem II. Suppose now the region covers but a portion of the globe. If, upon replacing the rest of the globe by a single country, we obtain a map of the type considered, he can do it also. We have but to apply the lemma to the ring of countries about the added country. By (A_3), no pair of the countries of this ring touch inside the ring. Therefor, picking out any two adjacent countries of the ring, A, B, we draw a line from one to the other, passing through every country the man wishes to visit. We now join the two ends of this line, completing the man's path.

More generally, whenever the conditions of the lemma are satisfied by the ring, calling some two adjacent countries A and B, we obtain the desired path.

4. Further remarks.

Necessity of (A_3). Theorem I would not be true if the assumption that there are no circuits of three edges other than the elementary triangles were omitted. That is, Theorem II would not be true if the assumption (A_3) were omitted. The following example shows this.[3]

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[3] This example of such a map containing the least number of regions was communicated to me by C. N. Reynolds.

The number P_n. In constructing the normal form for a graph, we divide an n-sided polygon into triangles by diagonals. It is interesting to know in how many ways we can do this. The formula for this number was found by Euler. A simple proof was first given by Lamé:[4]

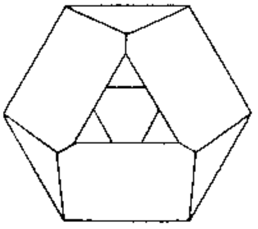

Fig. 9.

$$P_n = 2^{n-3} \frac{3 \cdot 5 \cdot 7 \cdots (2n-5)}{3 \cdot 4 \cdot 5 \cdots (n-1)}.$$

As we divide both the inside and outside of the polygon into triangles, we can construct in this manner P_n^2 different figures. Of course these are not all graphs of the type considered, and many of them give the same graph. For instance, there are 96 different circuits in the graph, Fig. 1.

[4] J. Math. Pures Appl. (1), 3 (1838), pp. 505–507.

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