

**HOMEWORK 8 (18.315, FALL 2005)**

1) Give a direct combinatorial proof of the hook-length formula for the number of standard Young tableaux of shape  $(n - 2k, 2, \dots, 2)$ ,  $k$  times.

2) Fix a tree  $t$  with  $n$  vertices and root  $R$ . Let  $S(v)$  denote the shortest path from  $v$  to  $R$ . Let  $\beta(v)$  be the number of vertices  $v'$  such that  $S(v) \subseteq S(v')$ . E.g.  $\beta(R) = n$ , and  $\beta(v) = 1$  for every leaf  $v \neq R$ . Denote by  $A(t)$  the set of bijections  $\gamma : t \rightarrow [n]$  such that:  $\gamma(v) < \gamma(v')$  for all vertices  $v, v' \in t$  with  $S(v) \subseteq S(v')$ ; e.g. this implies that  $\gamma(R) = 1$ . These bijections are called *increasing trees* of shape  $t$ . Prove that

$$|A(t)| = \frac{n!}{\prod_{v \in t} \beta(v)}.$$

3) Let  $Q$  be the set of partitions whose Young diagram is tileable by dominoes. Compute the generating function  $\sum_{\lambda \in Q} t^{|\lambda|}$ .

4) Let  $\Gamma(\lambda)$  be a graph on  $\text{SYT}(\lambda)$ , where two tableaux are connected by an edge if they differ at exactly two places. Prove that  $\Gamma(\lambda)$  is connected. Prove the analogue of this result for increasing trees of the same shape.

5) Let  $p_1, \dots, p_n$  be probabilities to be born on days  $1, \dots, n$  of the year (usually  $n = 365$ ). Let  $A$  be the event that  $k$  people (chosen independently) are all born on different days. Prove that  $\Pr(A)$  maximizes when  $p_i = 1/n$ .

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This Homework is due Friday December 9 at 11 am. (right after class).