

HOMEWORK 7 (18.315, FALL 2005)

- 1) Compute the number of permutations $\sigma \in S_n$ which are
 - a) both (123)- and (321)-avoiding
 - b) both (213)- and (312)-avoiding
 - c) both (123)- and (132)-avoiding
 - d) both (213)- and (231)-avoiding
 - e) both (213)- and (132)-avoiding

- 2) Let $\Sigma(k, \ell)$ be the set of permutations which are sortable with k queues and ℓ stacks. Prove or disprove: there exists a finite set of permutations $X(k, \ell)$ such that $\sigma \in S_n \cap \Sigma(k, \ell)$ if and only if σ avoids all permutations in X .

- 3) Consider only simple graphs (without loops or multiple edges). For a finite graph $G = (V, E)$, let $\alpha(G) = |E|/\binom{|V|}{2}$. For an infinite graph G let

$$\alpha_n(G) = \max_H \alpha(H),$$
 where the max is taken over all finite induced subgraphs of G with n vertices. Finally let $\alpha(G) = \limsup \alpha_n(G)$ as $n \rightarrow \infty$.
 - a) Prove that every rational number $a \in [0, 1]$ is equal to $\alpha(G)$ for some finite G .
 - b) Prove that for infinite graphs $\alpha(G) \neq \frac{1}{3}$.
 - c) Prove that for infinite graphs $\alpha(G) \in \mathbb{Q}$.

- 4) Let P be a polygon. A billiard trajectory in P is a (possibly self-intersecting) polygon inscribed into P such that the angle of reflection equals the angle of incidence (see Figure 1). The polygons and trajectories can have finite or infinite number of intervals.
 - a) Prove that every acute triangle has a finite closed billiard trajectory. (*Hint*: search for those of length 3.)
 - b) Let P be an acute cone. Prove that every billiard trajectory has a finite number of intervals (some of them going to infinity).

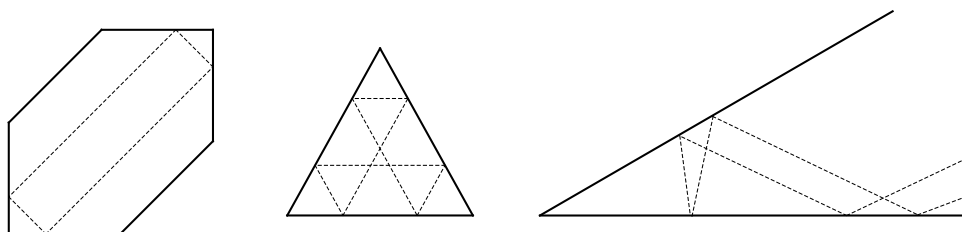


FIGURE 1. Three billiard trajectories.

This Homework is due Wednesday November 23 at 4 pm. in my office (2-390) or by e-mail.