

**HOMEWORK 3 (18.315, FALL 2005)**

- 1) Decide whether a rectangle  $[50 \times 60]$  can be tiled with rectangles
  - a)  $[20 \times 15]$       b)  $[5 \times 8]$
  - c)  $[6.25 \times 15]$     d)  $[2 - \sqrt{2} \times 2 + \sqrt{2}]$
- e) Find and prove a general criterion for tileability of a rectangle  $[a \times b]$  with rectangular tiles  $[p \times q]$ .
- 2) Let  $u_n$  be the number of *alternating permutations*  $\sigma \in S_n$ , i.e. permutations with  $\sigma(1) < \sigma(2) > \sigma(3) < \dots$ . Prove that the circled numbers in the following Pascal-style triangle are  $u_n$ . Here each number is the sum of two: one from above and one in the same row in the direction of 0.

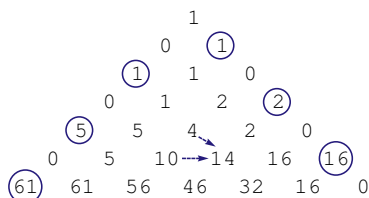


FIGURE 1. Triangle to compute numbers  $u_n$ .

- 3) Let  $T_n(x, y)$  be the Tutte polynomial of  $K_n$ . Prove that  $u_n = |T_{n+1}(1, -1)|$ .
- 4) In a spanning tree  $t \in K_n$  we say that vertices  $i$  and  $j$  form an *inversion* if  $i < j$  and  $j$  lies on the shortest path from  $i$  to 1. Let  $\text{inv}(t)$  be the number of inversions in  $t$ . Define

$$f_n(q) = \sum_{t \in K_n} q^{\text{inv}(t)}$$

Express  $f_n(q)$  via  $T_n(1, y)$ .

- 5) Let  $P_n$  be a polytope in  $\mathbb{R}^d$  defined by inequalities  $x_i \geq 0, 1 \leq i \leq n$ , and

$$x_i + x_{i+1} \leq 1, \quad 1 \leq i < n.$$

- a) Compute the number of integer points in  $P_n$   
(*Hint*: find a classical combinatorial interpretation).
- b) Compute the volume of  $P_n$   
(*Hint*: find a combinatorial interpretation in terms of  $u_n$ .)
- c) Give a combinatorial interpretation for the number of integer points in  $k \cdot P_n$ , generalizing part b). Here  $k \cdot X = \{k \cdot x \mid x \in X\}$ , and  $k \in \mathbb{N}$ .
- 6) Ex. 70 on p. 177 in *Bollobas*, MGT.

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This Homework is due on Wednesday October 12 at 4 pm. in my office (2-390) or by e-mail. Please remember to write the name(s) of your collaborators (see collaboration policy).