L4 (IUM, Spring)

March 2, 2021

Convex Polytopes

Dehn invariant ✓

elementary moves $2 \leftrightarrow 1$

<u>Th</u> (Sydler) P, Q – convex polytopes in $\mathbb{R}^3$

$vol(P) = vol(Q)$

$\rightarrow \varphi_f(P) = \varphi_f(Q)$ $\quad\forall$ Kagan function f

<u>Then</u> $\quad P \sim Q$

<u>Open problem</u>

Does this hold $\mathbb{S}^3$, $\mathbb{H}^3$?

vol, Dehn inv $\leftarrow$ ✓

<u>Simpler open problem</u>

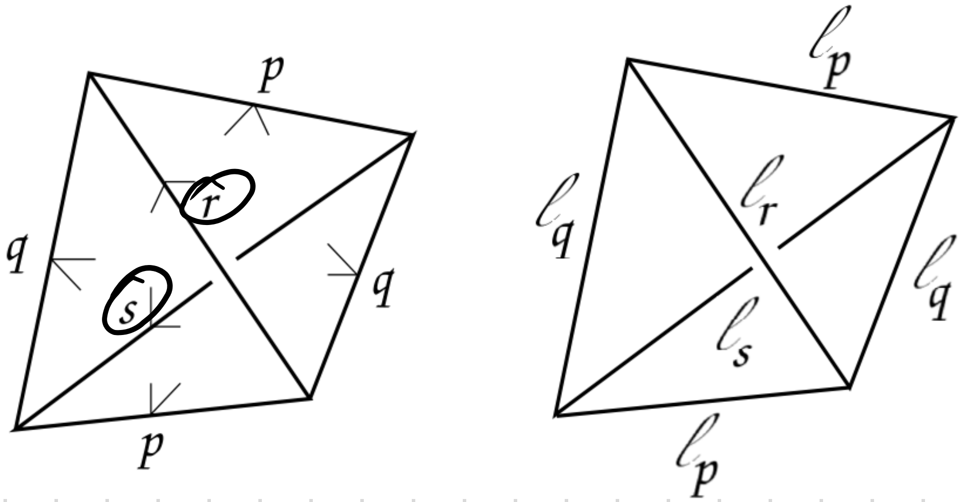
$\Delta_1, \Delta_2 \subset \mathbb{S}^3$, $\operatorname{vol} \Delta_1 = \operatorname{vol} \Delta_2$

and all dihedral angles are

rational $\in \pi \mathbb{Q} \Rightarrow \Delta_1 \sim \Delta_2$???

SPHERICAL TETRAHEDRA WITH RATIONAL VOLUME, AND SPHERICAL PYTHAGOREAN TRIPLES

ALEXANDER KOLPAKOV AND SINAI ROBINS

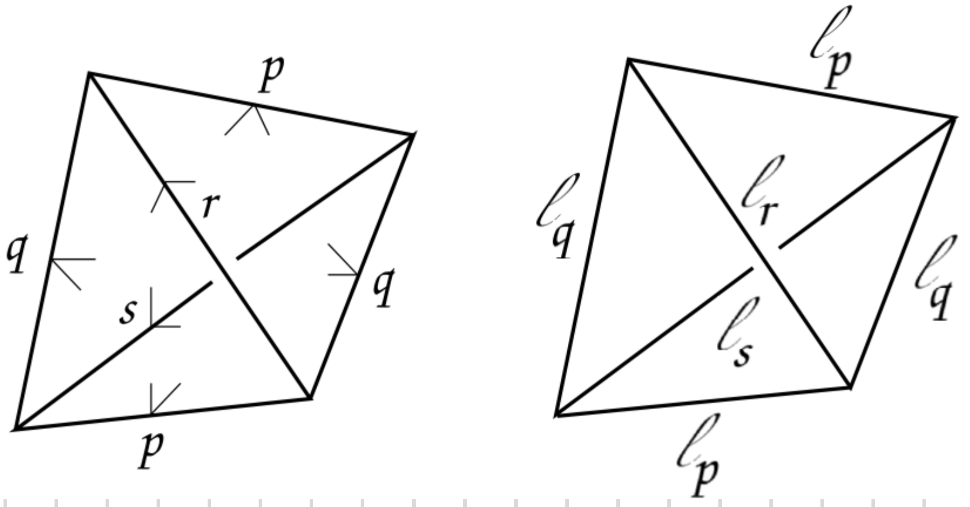


$$\cos p \cdot \cos q + \cos r = 0,$$

$$(p, q, r, s) = \left(\frac{5}{18}\,\pi, \frac{2}{9}\,\pi, \frac{13}{18}\,\pi, \frac{11}{18}\,\pi\right)$$

$$(\ell_p, \ell_q, \ell_r, \ell_s) = \left(\frac{5}{18}\,\pi, \frac{2}{9}\,\pi, \frac{5}{18}\,\pi, \frac{7}{18}\,\pi\right)$$

$$\operatorname{vol} T = \pi^2/162.$$



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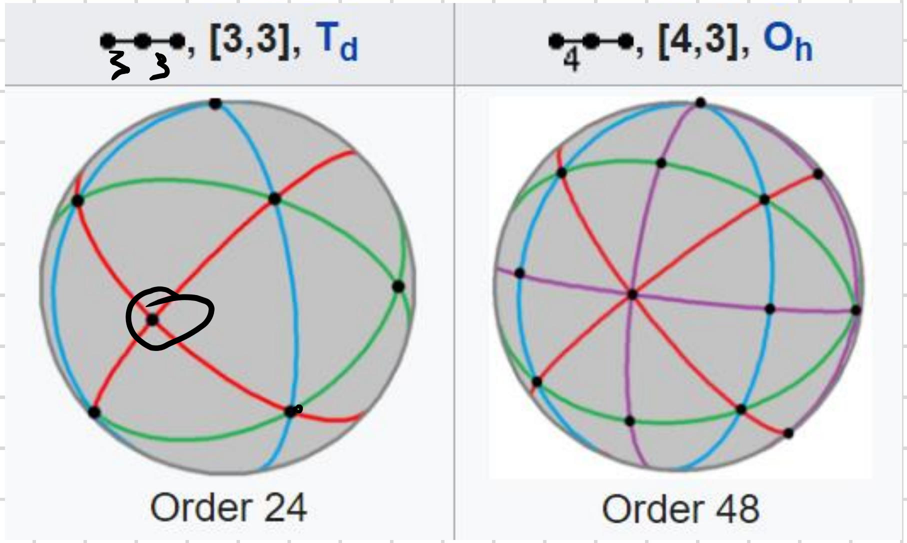
$$\mathrm{vol}\, T = \pi^2/162.$$

$\underline{Q}$ $T \sim T'$
???



SS^2

Coxeter tetrahedra in $\mathbb{S}^3$

Symbol	Coxeter diagram	Volume
$I_2(k) \times I_2(l)$		$\frac{\pi^2}{2kl}$

T' $k = l = 9 \Rightarrow \mathrm{vol} = \frac{\pi^2}{162}$

$/\underline{p = r = s = \frac{\pi}{2}}, \; q = \frac{\pi}{9}/$

Monge maps

Def $P, Q \subset \mathbb{R}^d$ – convex polytopes

$\varphi : P \to Q$ ← Monge map if

1) φ is PL ←
2) φ is continuous
3) φ is vol-preserving ←

Monge vs scissors-cong.

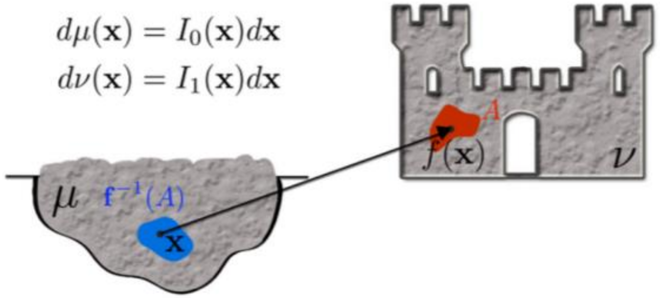
Th $vol(P) = vol(Q) \Rightarrow \exists\, \varphi : P \to Q$ s.t. φ is Monge map

What is Optimal Transport?

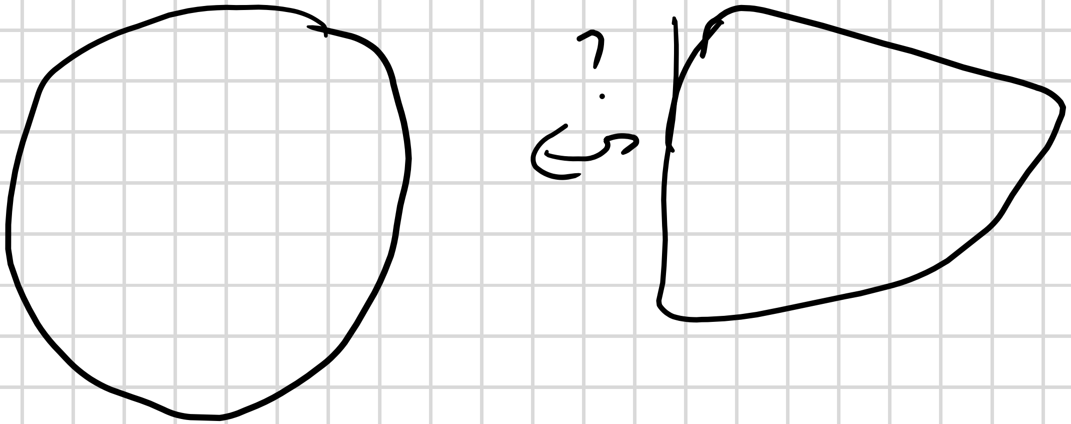
- The problem was originally studied by Gaspard Monge in the 18'th century.

Gaspard Monge
1746-1818

$$d\mu(\mathbf{x}) = I_0(\mathbf{x})d\mathbf{x}$$
$$d\nu(\mathbf{x}) = I_1(\mathbf{x})d\mathbf{x}$$



Le mémoire sur les déblais et les remblais
(The note on land excavation and infill)



orthog vs. contin.

Def
$P \bowtie Q$ ← Monge equiv.

$\exists\ \varphi : P \to Q$

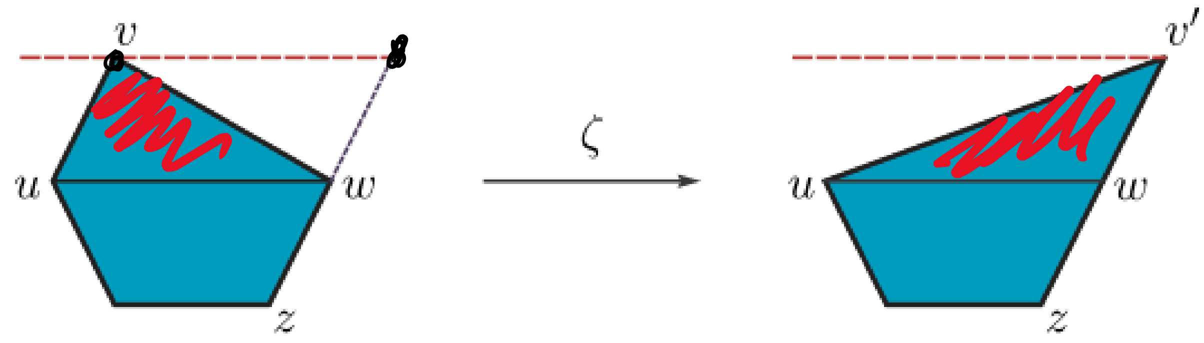


FIGURE 18.1. Monge map $\zeta : P \to P'$.

Proof of thm **$d=2$**

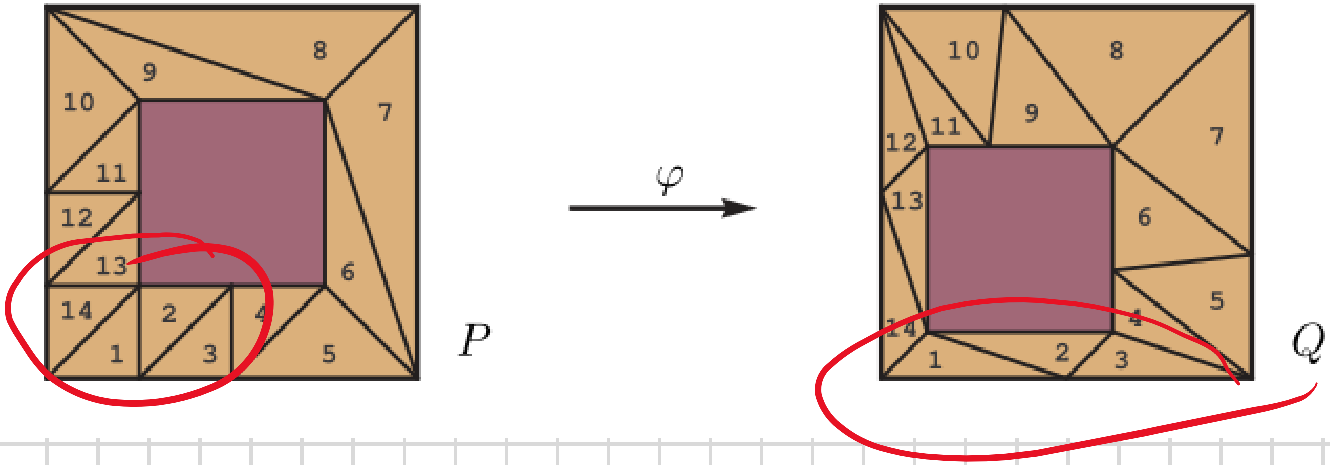
$P \subset \mathbb{R}^2$

n-gon

$$P \bowtie \Delta \,, \qquad Q \bowtie \Delta' \implies P \bowtie Q$$

$\square$

Q: in $\mathbb{R}^3$?



monge map

$\varphi : P \to Q$

Prop $P, Q \subset \mathbb{R}^d$ convex polytopes

$\Rightarrow \exists\ \varphi : P \to Q$ s.t. φ - 1), 3)

1) → PL

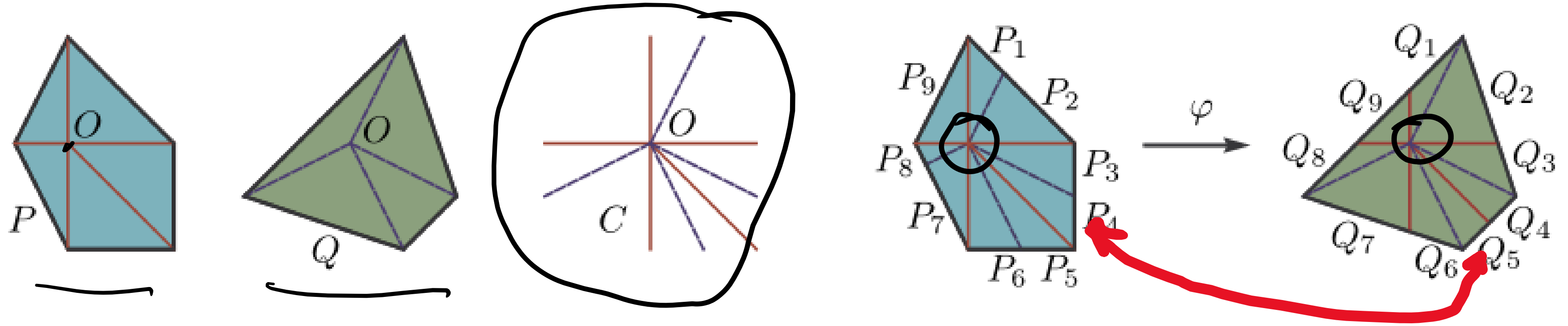
2) — vol-pres.

$\mathrm{vol}(P) = \mathrm{vol}(Q) = 1$

D $P = \bigsqcup \Delta_i$, $Q = \bigsqcup \Delta_i'$

$\mathrm{vol}\,\Delta_i = \alpha_i$ $\quad \Delta_i = \bigsqcup \Delta_{ij} \leftarrow \alpha_i \beta_j$

$\mathrm{vol}\,\Delta_i' = \beta_j$ $\quad \Delta_j' = \bigsqcup \Delta_{ij}'$ —||—



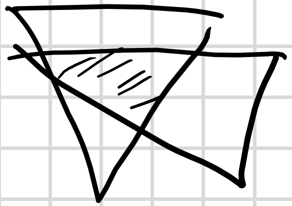
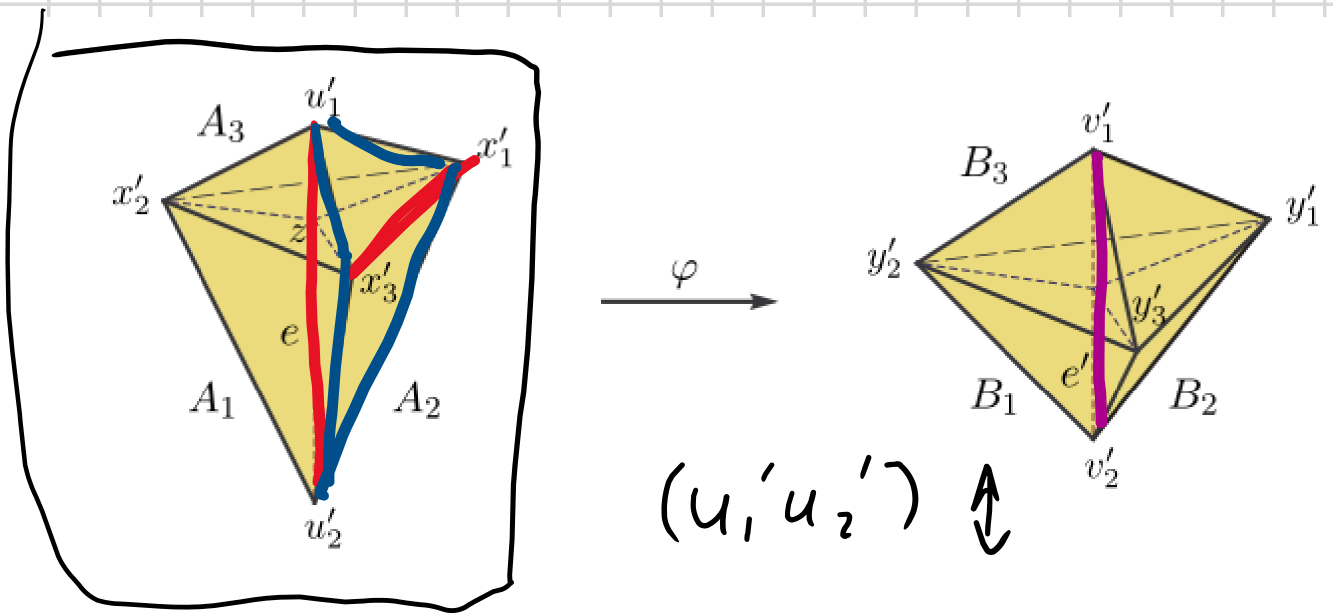


FIGURE 18.2. Polygons P, Q with fans F, G, the union fan $C = \widetilde{C}$, and the continuous PL-map $\varphi : P \to Q$.

Lem| $P, Q \subset \mathbb{R}^d$ - conv pol.

$\exists\, \varphi : P \to Q$ s.t. cont, PL

D Fans for P and Q $\square$



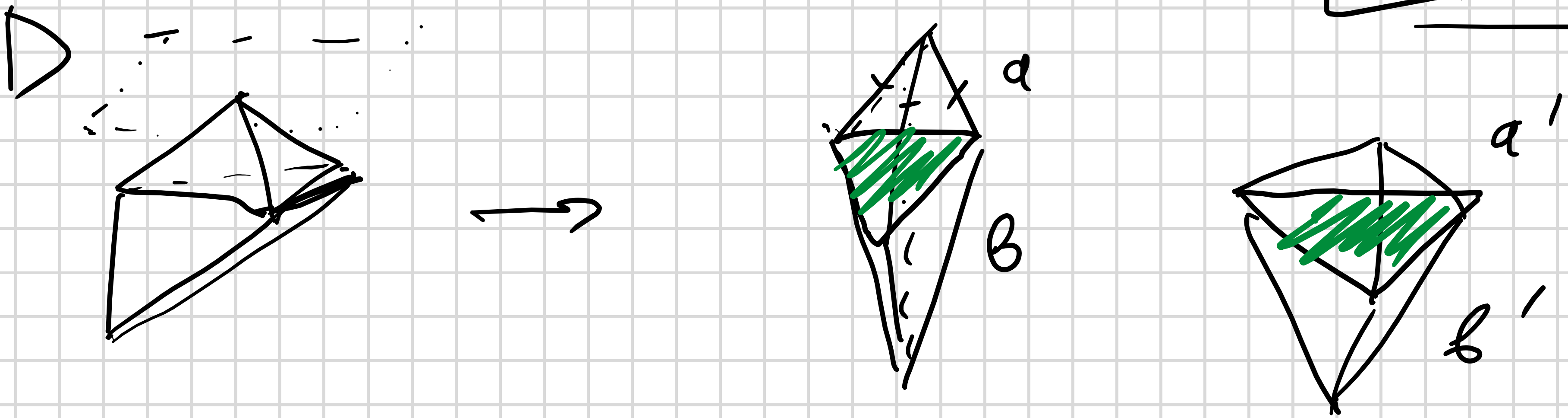
$(u_1' u_2') \updownarrow$

FIGURE 18.3. The map $\varphi : P' \to Q'$, where $P = A_1 \cup A_2 \cup A_3$ and $Q' = B_1 \cup B_2 \cup B_3$.

≤ 2 Th holds for bipyramids

simplex $\leftarrow d+1$

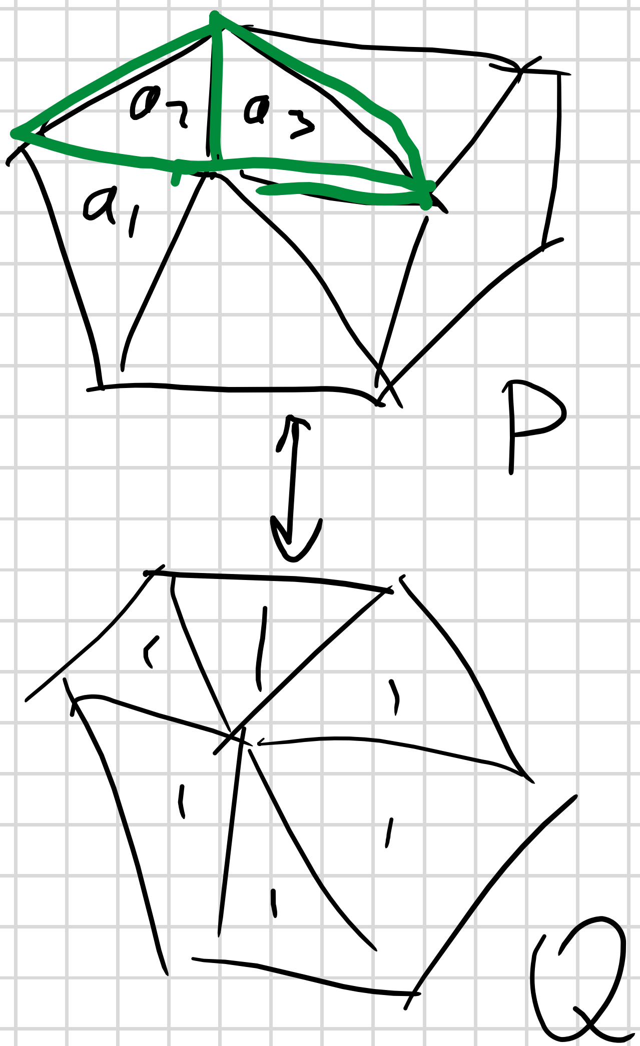
b. pyramid $\leftarrow d+2$ vertices



$a+b = a'+b'$

$\blacksquare$

Proof of Th



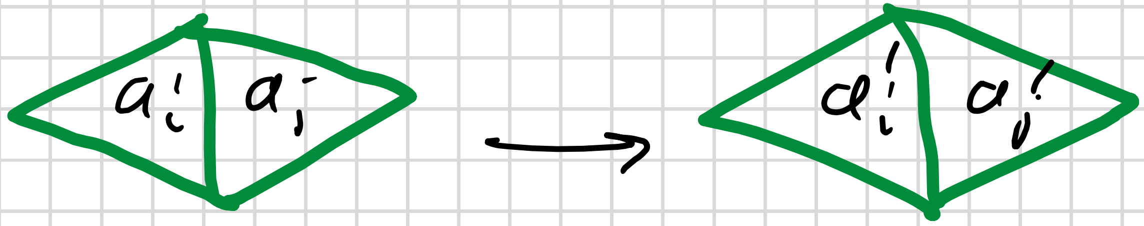
Take $\varphi : P \to Q$ from L1

$a_i \in \mathbb{R}_+$ $\qquad \varphi : \Delta_i \to \Delta_i'$

$P = \bigsqcup \Delta_i$

$Q = \bigsqcup \Delta_i'$

$\mathrm{vol}\, \Delta_i' = a_i \, \mathrm{vol}\, \Delta_i$



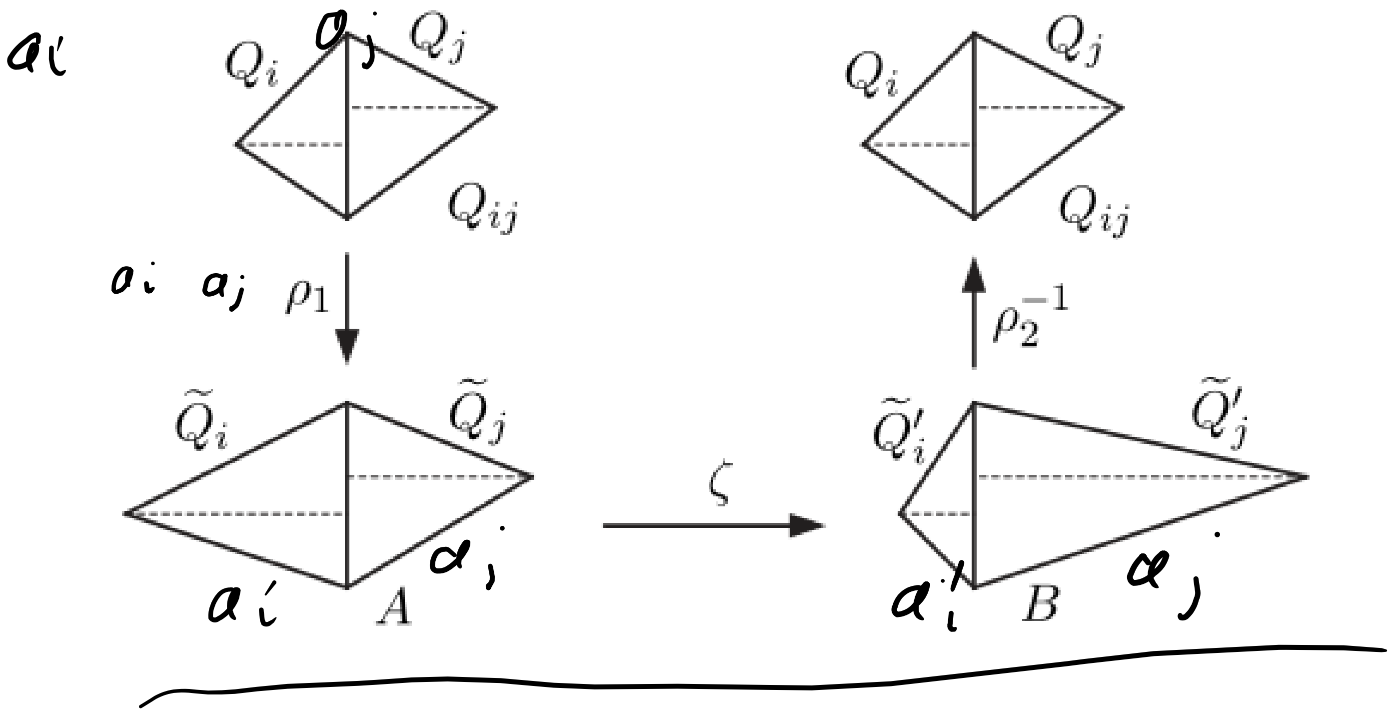
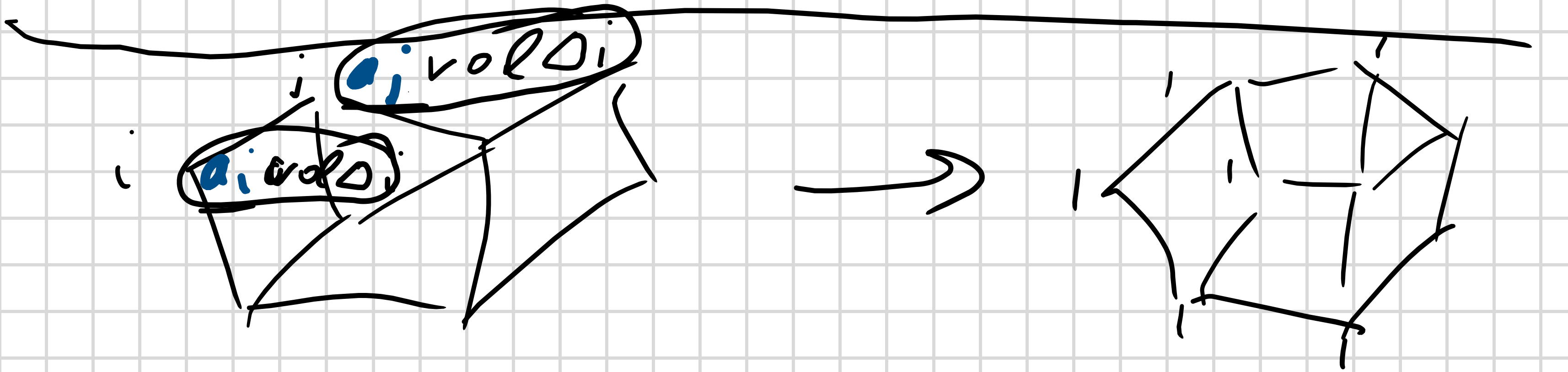


FIGURE 18.4. Map $\gamma : Q_{ij} \to Q_{ij}$, where $\gamma = \rho_2^{-1} \circ \zeta \circ \rho_1$.

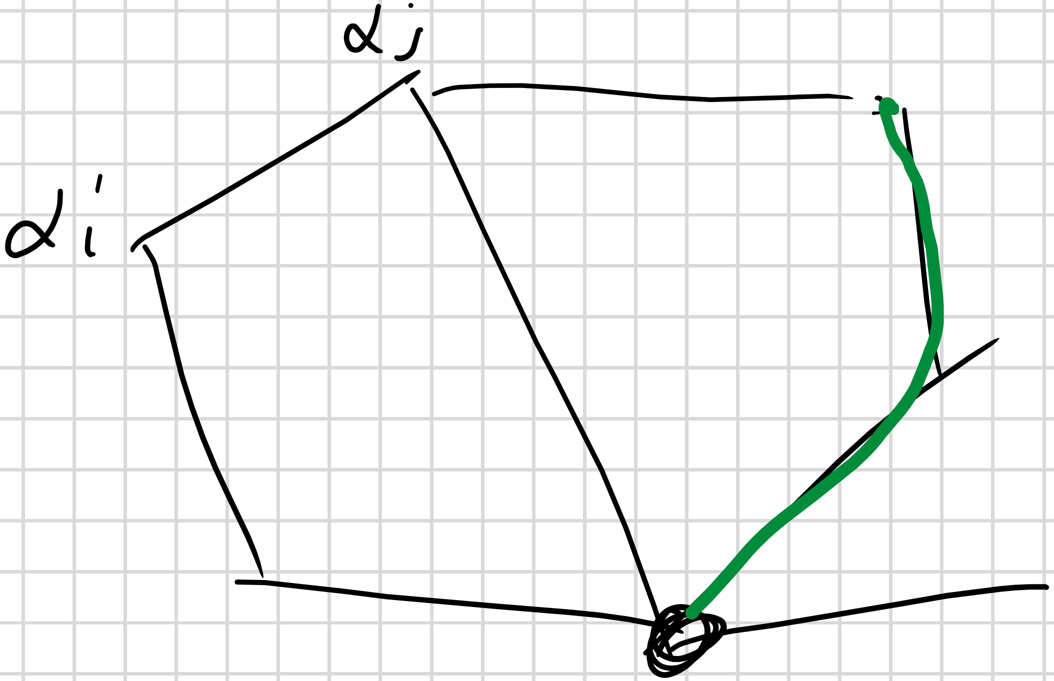
$$\left\{ \begin{array}{l} a_i \operatorname{vol} \Delta_i + \\ a_j \operatorname{vol} \Delta_j = \\ = a_i' \operatorname{vol} \Delta_i' \\ + a_j' \operatorname{vol} \Delta_j' \end{array} \right\}$$

$\varphi : P \to Q$

$\zeta \circ \varphi$



Kuperberg



Γ – connected graph

$(\alpha_i \ \alpha_j) \to (\alpha_i' \ \alpha_j')$

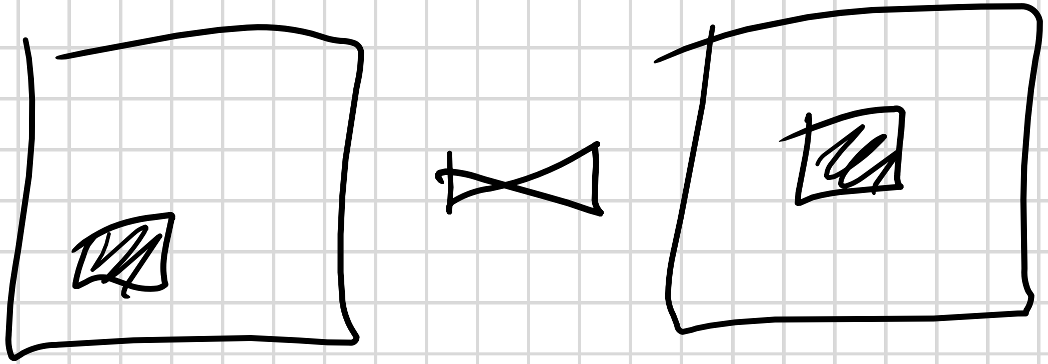
$\alpha_i + \alpha_j = \alpha_i' + \alpha_j'$

Th $M, N \leftarrow$ PL-manifolds

$M \simeq N \leftarrow$ PL-homeom

$\operatorname{vol} M = \operatorname{vol} N$

$\Rightarrow \exists$ vol-pres PL-homeom.

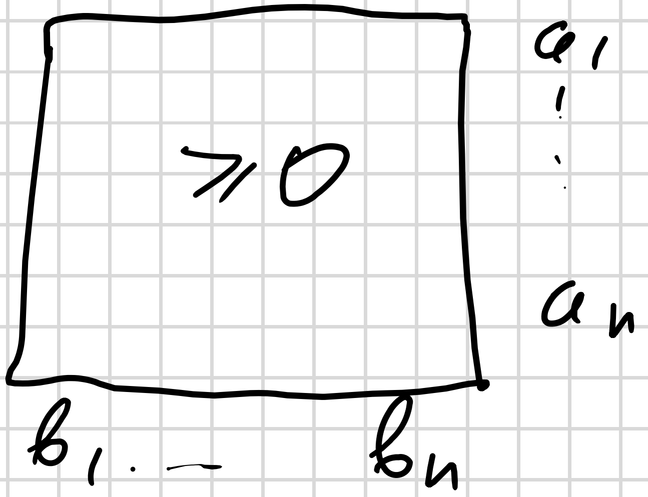
$\Rightarrow$ 

Exc

Example of Monge Map

Two polytopes $\bar{a} = (a_1, \ldots a_n)$, $\bar{b} = (b_1, \ldots b_n)$ } fixed

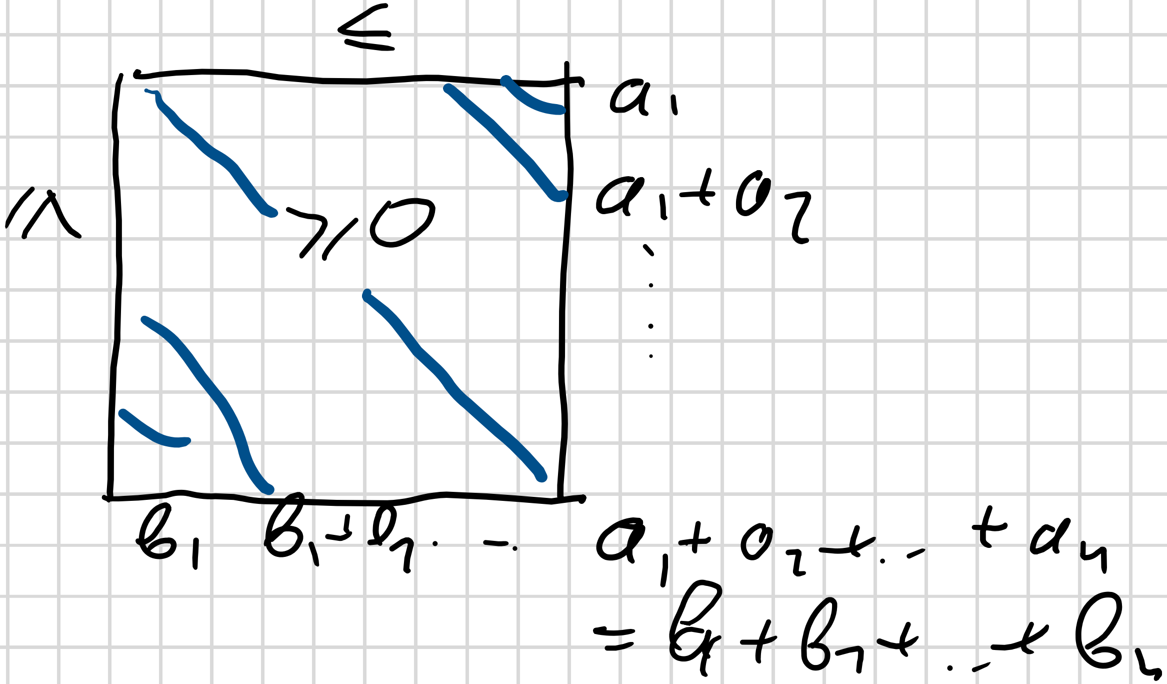
P ← transportation polytope.



$$P_{ab} = \left\{ (x_{ij}) \;,\; x_{ij} \geq 0 \quad \forall i,j \quad \sum_{j=1}^{n} x_{ij} = a_i \quad \sum_{i=1}^{n} x_{ij} = b_j \right\}$$

$$\boxed{a_1 + \ldots + a_n = b_1 + \ldots + b_n}$$

$\underline{Q}$ $\leftarrow$ plane partition polytope



$$Q_{\bar a \bar b} = \left\{ \left(y_{ij} \right)_{1 \le i,j \le n} \quad y_{ij} \ge 0 \right.$$

$\oplus$ diagonal sums condition $\}$

Th1 $\text{vol } P_{\bar a \bar b} = \text{vol } Q_{\bar a \bar b} \quad \forall\, \bar a, \bar b$

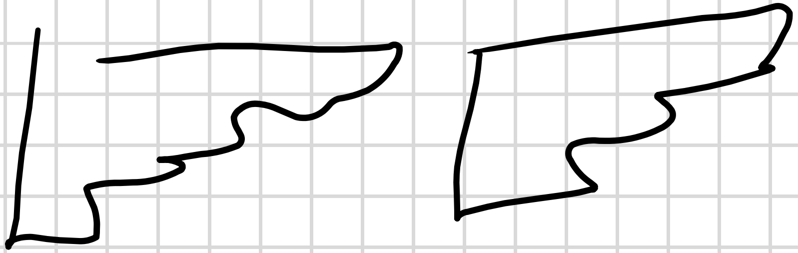
Th2 # int points in P_{ab} = # int. points in $Q_{\bar a \bar b}$

$\bar{b}, \bar{a} \in \mathbb{N}^n$

\# int pts in $P_{\bar{a}\bar{b}}$ = # int matrices (row sums $a_1, \dots, a_n$; column sums $b_1, \dots, b_n$)

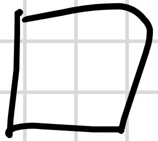
$\|$

\# —— / L $Q_{\bar{a}\bar{b}}$ = # pairs



SSYT$(\lambda, \bar{a})$ SSYT$(\lambda, \bar{b})$

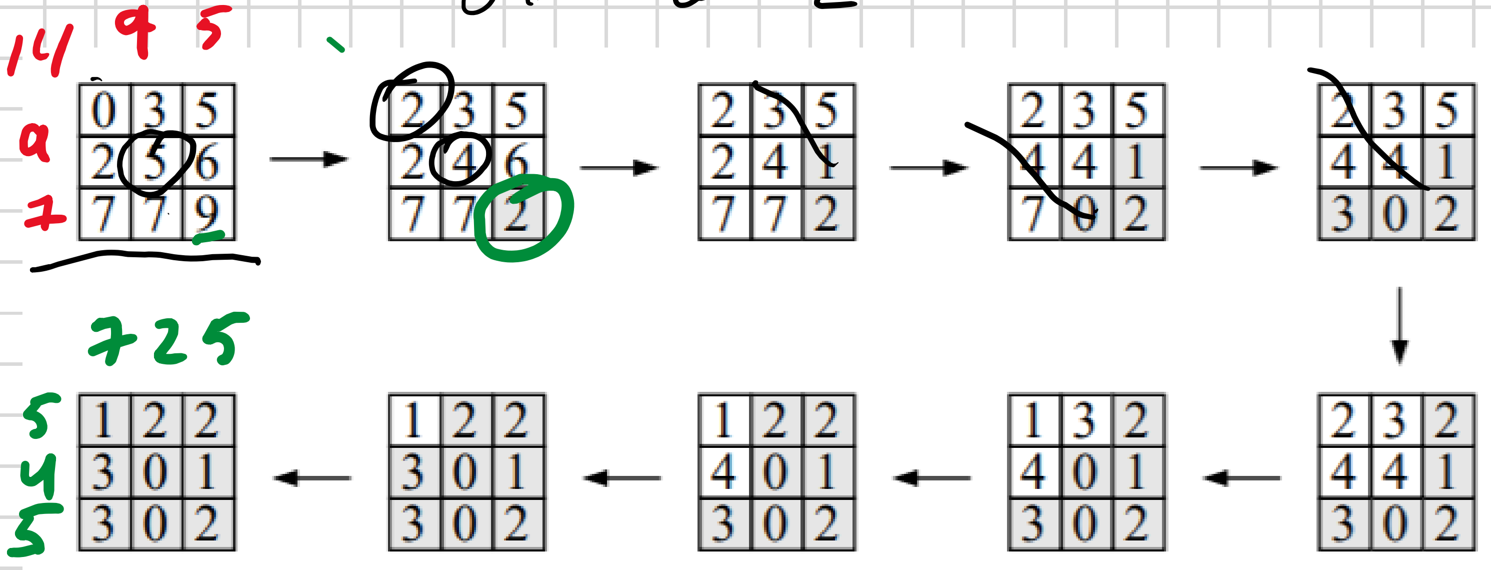
Th RSK (Knuth, 1980)

 $\longleftrightarrow$ ()

Now : explicit construction of Monge map $\Phi : P_{\bar{a}\bar{b}} \to Q_{\bar{a}\bar{b}}$

Important : $\Phi = RSK$

$0+2-3=2$

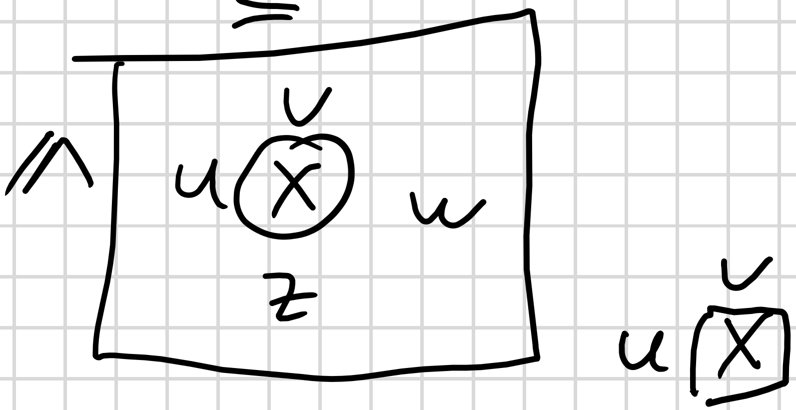


$\Phi^{-1}: Q \to P$

$\bar{a} = (5\ 4\ 5)$

$b = (7\ 2\ 5)$

$5 \to (3+6-5) = 4$



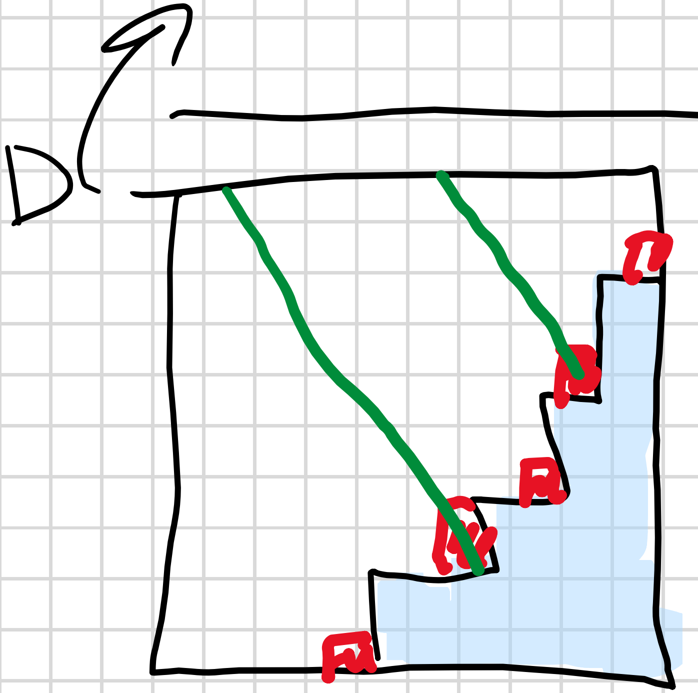
$X \to \underline{\max\{u, v\} + \min\{w, z\} - X}$

$\underline{X - \max\{u, v\}} \to \square$

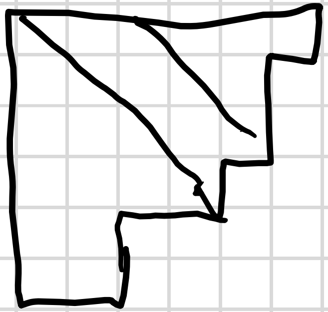
L $\Phi : Q_{\bar a \bar b} \to P_{\bar a \bar b}$ — PL, vol-pres and continuous.

b_{ij}

Prop: Φ does NOT depend on the order of squares removed.



commutativity of min-max maps



+ L

∎