HOME ASSIGNMENT 2 (MATH 285N, SPRING 2010)

1. Let $P \subset \mathbb{R}^3$ be a convex polytope. Suppose the faces of P are colored in red and blue such that no two red faces are adjacent. Suppose the total area of red faces is larger than the total area of blue faces. Prove that P is not circumscribed around a sphere. Find an explicit example of such polytope P.

2. Let $P, Q \subset \mathbb{R}^2$ be two convex polygons. Prove that the union $R = P \cup Q$ is convex if and only if every interval $[vw] \subset R$, for for every pair of vertices v of P and w of Q. Does this hold in $\mathbb{R}^3$ as well?

3. Let $P \subset \mathbb{R}^3$ be a convex polytope of volume 1. Suppose on every face F in P, there is a slow ant which moves clockwise along the edges of F with the speed at least 0.0001. Prove that at some point in the future, some two ants will meet.

4. Two polygons $P, Q \subset \mathbb{R}^2$ are called *Z-congruent* if P can be decomposed into triangles which can then rearranged to form Q, such that the points on the boundary of P are mapped into the boundary of Q. Prove that every two polygons with the same area and perimeter are Z-congruent.

5. A *hyper-cross* in $\mathbb{R}^d$ is a union of $2d + 1$ unit cubes in the grid $\mathbb{Z}^d$, all attached to a single central cube. Prove that hyper-crosses can tile the whole space $\mathbb{R}^d$.

6. Prove that the regular cross-polytope $Q \subset \mathbb{R}^4$ can tile the space $\mathbb{R}^4$ periodically. Use this to show that Q is scissor congruent to a 4-cube.

7. Suppose n particles in the space move at the same speed in different directions. Prove that all particles will eventually move into a convex position.