HOME ASSIGNMENT 1 (ENUMERATIVE COMBINATORICS, FALL 2010)

1. A coin with probability p of **H**, if flipped repeatedly. Use g.f.'s to compute
a) the expected time and the variance until **HH**;
b) the expected time until **HT**;
c) the expected time until either **HTH** or **THT**.
Bring the answer to a closed form. Give a high level explanation why these are rational functions in p.

2. Let $V = \mathbb{F}_q^n$. Use q-binomial coefficients to compute in a closed form:
a) For a given (k, ℓ), find the number of pairs of subspaces (U, W), such that $U, W \subset V$, $\dim U = k$, $\dim W = \ell$, and $U \cap W = \varnothing$.
b) For a given (k, ℓ, m), find the number of pairs of subspaces (U, W), such that $U, W \subset V$, $\dim U = k$, $\dim W = \ell$, and $\dim(U \cap W) = m$.
c) For a given (a_{ij}, b_i, c), find the number of triples of subspaces (U_1, U_2, U_3), such that $U_i \subset V$, $\dim(U_i \cap U_j) = a_{ij}$, $\dim U_i = b_i$, and $\dim(U_1 \cap U_2 \cap U_3) = c$.
d) Give a high level explanation why a closed formula in part *c*) cannot possibly be generalized to quadruples of subspaces.

3. Fix integers $k \ge \ell \ge 1$. Denote by $N(k, \ell)$ the number of sequences $(a_1, \ldots, a_{k+\ell})$ such that $a_i \in \{1, -1\}$ with exactly k 1's and ℓ -1's, and such that all partial sums $a_1 + \ldots + a_s \ge 0$, for all $1 \le s \le k + \ell$.
a) Check that $N(k, k)$ is the Catalan number.
b) Find a recurrence relation for the numbers $N(k, \ell)$, Pascal triangle style.
c) Find a closed formula for $N(k, \ell)$ and prove it using recurrence relations from *b*).
d) Give a bijective proof of the following identity:

$$N(n, n)^2 + N(n+1, n-1)^2 + \ldots + N(2n, 0)^2 \; = \; \frac{1}{2n+1}\binom{4n}{2n}.$$

4. *a*) Find the number of spanning trees in a $2 \times n$ grid graph. Use g.f. and then an explicit formula, Fibonacci numbers style. Can you express this number is terms of Fibonacci numbers?
b) Same question for a $2 \times n$ "cylinder", i.e. a product graph $K_2 \times C_n$, where C_n is a n-cycle.

5. Use Burnside's lemma to derive Jordan's theorem (1872) that if a group acts transitively, then it has a *derangement* (fixed point free element).

6. Consider the action of S_n on itself by conjugation. Check that the number of orbits is equal to the number of integer partitions $p(n)$. Use Burnside's lemma and explicit formulas for the sizes of conjugacy classes (see Stanley, 1.3.2) to obtain $p(n)$ as a summation. Is this a "formula" in Wilf's (rather weak) sense?

7. Prove that there exists a Fibonacci number which is divisible by 997. What about $F_n > 10^{100}$ which is 5 modulo 997?

8. Give a direct combinatorial proof that $p(n) \le F_n$ for all $n \ge 1$.

9. Use Euler's pentagonal theorem to show that $p(n)$ changes parity infinitely many times.

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This HA is due Fri Oct 8, before class. No late solutions will be accepted as I plan to present some solutions in class. Note that some of these problems are easier than others.

Collaboration is allowed and even encouraged, but each student has to write her/his own solution. Moreover, each study group is limited to three, and I ask you to write your collaborators on the top of the first page.