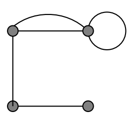
HOMEWORK 5 (MATH 61, SPRING 2017)

Solve: **RJ**, Sec 8.2 Ex 11, 12, 14, 15, 30, 32, 35, 36, 48, Sec 8.3 Ex 2, 4, 5, 7, 8, 14, 15, Sec 8.6 Ex 2, 3, 8, 9, 11, 12.

8.2

11. No such graph exists because the sum of the degrees of all vertices must be even.

12.
14. No such graph exists because the sum of the degrees of all vertices must be twice the number of edges.



15.
30. No Euler cycle because v_3 has odd degree
32. $a, b, c, h, i, c, d, b, f, e, d, i, e, j, f, g, j, h, a$
35. n is odd
36. m and n are even
48. K_2

8.3

2. $a, h, i, o, j, b, c, k, l, d, e, m, p, n, f, g, a$
4. Consider two subsets of vertices, $V_1 = \{d, e, g, h, n, l\}$ and $V_2 = \{c, j, m\}$. Each vertex in V_1 has degree 3 and is connected to exactly 2 vertices in V_2. Thus, any Hamiltonian cycle must contain exactly one edge connecting each vertex in V_1 with a vertex in V_2. This implies (a, c) and (m, o) are not in the cycle, which in turn makes (a, f) and (f, o) in the cycle. But this also implies (f, g) is not in the cycle which is not possible as only one of (g, c) and (g, m) is in the cycle.
5. Edges $(g, i,), (i, j), (j, g)$ must be in any Hamiltonian cycle.
7. $a, b, c, g, l, m, r, q, p, k, j, f, e, i, n, o, t, s, h, d, a$
8. We may remove one of parallel edges (a, e). Consider two subsets of vertices, $V_1 = \{b, d, g\}$ and $V_2 = \{c, e\}$. Each vertex in V_1 has degree 3 and is connected to exactly 2 vertices in V_2. Thus, any Hamiltonian cycle must contain at least one edge connecting each vertex in V_1 with a vertex in V_2. This is not possible as (a, c) and (a, e) must be in the cycle

14. Consider a sequence of all vertices of K_n, $v_1, v_2, \ldots, v_n$. Then $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle.
15. $m = n \geq 2$.

8.6

2. $(a, b, c, d, e, f) \to (1, 6, 4, 5, 2, 3)$
3. $(a, b, c, d, e) \to (2, 4, 5, 1, 3)$
8. G_1 has 8 edges, but G_2 has 9 edges.
9. G_2 contains a subgraph isomorphic to K_3 ($\{1, 2, 5\}$), but G_1 does not.
11. $(a, b, c, d, e, f, g, h, i, j, k, l) \to (1, 5, 6, 2, 3, 7, 8, 4, 9, 10, 11, 12)$
12. G_1 has a vertex with degree 2 (vertex c), but G_2 does not.

Problem I. Let $G = K_{6,6}$. Find the number of subgraphs of G isomorphic to W, where

a) $W = P_8$ $(\frac{6!}{2!})^2$

b) $W = P_9$ $\frac{(6!)^2}{2!}$

c) $W = C_8$ $\frac{1}{8}(\frac{6!}{2!})^2$

d) $W = C_9$ 0

e) $W = K_{3,4}$ $2\binom{6}{3}\binom{6}{4}$

f) $W = K_{4,4}$ $\binom{6}{4}^2$

g) $W = H_3$, the 3-cube. $\binom{6}{4}^2 4!$

Problem II. For each of these sequences, explain why they cannot be scores (degree sequences) of a simple graph.

a) (2,3,3,4,5,6,7)
The maximum degree of a simple graph is $|V| - 1$.

b) (0,1,2,3,4,5,6,7)
A vertex with degree $7 = |V| - 1$ must be connected to every other vertex. Thus, every vertex must have degree at least 1.

c) (3,3,3,5,5,5,5)
The sum of the degrees of all vertices must be even.

d) (1,2,3,4,4,6,6)
Two vertices with degree $6 = |V| - 1$ must be connected to every other vertex. Thus, every vertex must have degree at least 2.

e) (2,2,2,6,6,6,6)
Four vertices with degree $6 = |V| - 1$ must be connected to every other vertex. Thus, every vertex must have degree at least 4.

f) (2,3,4,5,5,7,7,7)
Three vertices with degree $7 = |V| - 1$ must be connected to every other vertex. Thus, every vertex must have degree at least 3.

Problem III. Prove that n-cube H_n has a Hamiltonian cycle, for all $n \geq 2$.

By induction: Base case ($n = 2$): H_2 has a Hamiltonian cycle. Inductive step: Assume H_n has a Hamiltonian cycle. In H_{n+1}, start with identical Hamiltonian cycles in each copy of H_n. Remove one edge from one of the cycles and the corresponding edge from the other cycle, and connect the endpoints of the removed edges to the corresponding vertices in the other copy of H_n. This is a Hamiltonian cycle on H_{n+1}.

Problem IV. Prove that $K_{n,n}$ has a Hamiltonian cycle, for all $n \geq 2$.

Let V and W be the two parts of $K_{n,n}$. Label the vertices of V $v_1, v_2, ..., v_n$ and the vertices in W $w_1, w_2, ..., w_n$. The edges connecting v_i to w_i for $i \in \{1, 2, ..., n\}$, connecting v_i to w_{i+1} for $i \in \{1, 2, ..., n-1\}$ and connecting v_n to w_1, form a Hamiltonian cycle.

Problem V. Decide whether the following graphs are isomorphic or non-isomorphic. We include just the answers for a) and b) for the brevity and typesetting reasons – for a complete proof you need to include the isomorphism that can be found by trial and error.

a) isomorphic

b) isomorphic

c) non-isomorphic (first graph has a C_3, the other does not).