HOMEWORK 1 (MATH 61, SPRING 2017)

Solve: **RJ**, Sec. 2.2 Ex 28, 29, Sec. 2.4 Ex 2, 3, 6, 9, Sec 3.2 Ex 7, 9, 13, 14.

2.2

28. Suppose that there exist positive integers m, n such that $m^3+2n^2=36$. Then $m^3<36$, and thus $m<(36)^{\frac{1}{3}}<4$. Since both $2n^2$ and 36 are even, m^3 must be even, so is m. Thus, $m=2$. But this implies $n^2=36-2^3=28$ which is not a square of any positive integer, which is a contradiction.
29. Suppose that there exist positive integers m, n such that $2m^2+4n^2-1=2(m+n)$. Then the left hand side must be odd, but the right hand side is even.

2.4

2. Base case: $1\cdot 2=2=\frac{1\cdot 2\cdot 3}{3}$.
Inductive step: Suppose $1\cdot 2+2\cdot 3+\ldots+n(n+1)=\frac{n(n+1)(n+2)}{3}$ for $n\geq 1$. We want to show $1\cdot 2+2\cdot 3+\ldots+(n+1)(n+2)=\frac{(n+1)(n+2)(n+3)}{3}$. By the inductive assumption, we have
$$\begin{aligned}1\cdot 2+2\cdot 3+\ldots+(n+1)(n+2)&=\frac{n(n+1)(n+2)}{3}+(n+1)(n+2)\\&=(n+1)(n+2)\left(\frac{n}{3}+1\right)\\&=\frac{(n+1)(n+2)(n+3)}{3}.\end{aligned}$$

3. Base case: $1(1!)=1=2!?1$.
Inductive step: Suppose $1(1!)+2(2!)+\ldots+n(n!)=(n+1)!-1$ for $n\geq 1$. We want to show $1(1!)+2(2!)+\ldots+(n+1)((n+1)!)=(n+2)!-1$. By the inductive assumption,
$$\begin{aligned}1(1!)+2(2!)+\ldots+(n+1)((n+1)!)&=(n+1)!-1+(n+1)((n+1)!)\\&=(n+1+1)((n+1)!)-1=(n+2)!-1.\end{aligned}$$

6. Base case: $1^3=1=\left(\frac{1\cdot 2}{2}\right)^2$.
Inductive step: Suppose $1^3+2^3+\ldots+n^3=\left(\frac{n(n+1)}{2}\right)^2$ for $n\geq 1$. We want to show $1^3+2^3+\ldots+(n+1)^3=\left(\frac{(n+1)(n+2)}{2}\right)^2$. By the inductive assumption,
$$\begin{aligned}1^3+2^3+\ldots+(n+1)^3&=\left(\frac{n(n+1)}{2}\right)^2+(n+1)^3\\&=(n+1)^2\left(\frac{n^2}{4}+(n+1)\right)\\&=(n+1)^2\left(\frac{n^2+4n+4}{4}\right)\\&=(n+1)^2\left(\frac{n+2}{2}\right)^2=\left(\frac{(n+1)(n+2)}{2}\right)^2.\end{aligned}$$

9. Base case: $\frac{1}{2^2-1}=\frac{1}{3}=\frac{3}{4}-\frac{1}{2(2)}-\frac{1}{2(3)}$.
Inductive step: Suppose $\frac{1}{2^2-1}+\frac{1}{3^2-1}+\ldots+\frac{1}{(n+1)^2-1}=\frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}$ for $n\geq 1$.

We want to show $\frac{1}{2^2-1}+\frac{1}{3^2-1}+\ldots+\frac{1}{(n+2)^2-1}=\frac{3}{4}-\frac{1}{2(n+2)}-\frac{1}{2(n+3)}$. By the inductive assumption,

$$\begin{aligned}
\frac{1}{2^2-1}+\frac{1}{3^2-1}+\ldots+\frac{1}{(n+2)^2-1} &= \frac{3}{4}-\frac{1}{2(n+1)}-\frac{1}{2(n+2)}+\frac{1}{(n+2)^2-1}\\
&= \frac{3}{4}-\frac{1}{2(n+2)}+\frac{1}{n^2+4n+3}-\frac{1}{2(n+1)}\\
&= \frac{3}{4}-\frac{1}{2(n+2)}+\frac{1}{(n+3)(n+1)}-\frac{1}{2(n+1)}\\
&= \frac{3}{4}-\frac{1}{2(n+2)}+\frac{1}{n+1}\left(\frac{1}{n+3}-\frac{1}{2}\right)\\
&= \frac{3}{4}-\frac{1}{2(n+2)}+\frac{1}{n+1}\cdot\frac{2-(n+3)}{2(n+3)}\\
&= \frac{3}{4}-\frac{1}{2(n+2)}-\frac{1}{n+1}\cdot\frac{n+1}{2(n+3)}\\
&= \frac{3}{4}-\frac{1}{2(n+2)}-\frac{1}{2(n+3)}.
\end{aligned}$$

3.2

7. $t_{2077}=2(2077)-1=4153$.
9. Note t_n is the nth odd number. The sum of the first n odd numbers is n^2. So,
13. Yes.
14. No.

I. a) In the base case $n=1$, we interpret the left hand side as having no factors since the first factor in the expression is $(1-\frac{1}{2^2})$ which is the last factor in the case $n=2$. This is a common mathematical convention. The product of no factors is by convention 1. The right hand side in the case $n=1$ is also $\frac{1+1}{2\cdot 1}=1$, as needed.

In the induction step we assume $(1-\frac{1}{2^2})\cdots(1-\frac{1}{n^2})=\frac{n+1}{2n}$. Then,

$$\begin{aligned}
(1-\frac{1}{2^2})\cdots(1-\frac{1}{n^2})(1-\frac{1}{(n+1)^2}) &= [(1-\frac{1}{2^2})\cdots(1-\frac{1}{n^2})](1-\frac{1}{(n+1)^2})\\
&= \frac{n+1}{2n}(1-\frac{1}{(n+1)^2})\\
&= \frac{n+1}{2n}\cdot\frac{(n+1)^2-1}{(n+1)^2}\\
&= \frac{n+1}{2n}\cdot\frac{((n+1)-1)((n+1)+1)}{(n+1)^2}\\
&= \frac{(n+1)+1}{2(n+1)}
\end{aligned}$$

using the induction hypothesis in the second equality.

b) In the base case $n=1$, the left hand side, $1^3=1$ and the right hand side, $(1)^2=1$ agree. In the induction step we assume $1^3+\ldots+n^3=(1+\ldots+n)^2$ for a particular natural number n. Recall from class that for every natural number k, $1+2+\ldots+k=\frac{k(k+1)}{2}$. Then using this fact for $k=n$ in the third equality and $k=n+1$ in the fifth equality as well as the induction hypothesis in the second equality,

$$\begin{aligned}1^3+\ldots+n^3+(n+1)^3 &= [1^3+\ldots+n^3]+(n+1)^3\\ &= (1+2+\ldots+n)^2+(n+1)^3\\ &= (\frac{n(n+1)}{2})^2+(n+1)^3\\ &= (n+1)^2(\frac{n^2}{4}+n+1)\\ &= (n+1)^2(\frac{n^2+4n+4}{4})\\ &= \frac{(n+1)^2(n+2)^2}{4}\\ &= (\frac{(n+1)(n+2)}{2})^2.\end{aligned}$$

II. [One of many solutions.] Let $a_1 = a_2 = 1$ and $a_{n+1} = a_n - a_{n-1}$. This defines a unique sequence $\{a_n\}$. Then $a_3 = 0$, $a_4 = -1$, $a_5 = -1$, $a_6 = 0$, $a_7 = 1$, and $a_8 = 1$. So if we define another sequence $b_n := a_{n+6}$, then $b_1 = a_7 = b_2 = a_8 = 1$ and $b_{n+1} = a_{n+7} = a_{n+6} - a_{n+5} = b_n - b_{n-1}$ (when $n > 1$). Therefore the sequence $\{b_n\}$ meets the defining condition of the $\{a_n\}$ and so $\{b_n\} = \{a_n\}$. This means every natural number n, we have $a_{n+6} = b_n = a_n$.

(The following can be applied to all repeating sequences. Replace 6 with the period) We can show by induction on a natural number q that for any natural number r we have $a_{r+6q} = a_r$. We have already proven the first paragraph the base case. In the induction step we assume $a_{r+6q} = a_r$ and using this and the base case for $r + 6q$ we have $a_{r+6(q+1)} = a_{(r+6q)+6} = a_{r+6q} = a_r$. This completes the induction.

Now for any natural number n, we can divide n by 6 and use this to write $n = r + 6q$ where $r \in \{1, 2, 3, 4, 5, 6\}$ (r is the remainder when dividing n by 6 unless the remainder is 0, in which case $r = 6$). Then $a_n = a_r \in \{1, 0, -1\}$ since we computed the first six terms of the sequence. In particular, $-3 \leq a_n \leq 3$.

III. Find closed formulas for elements in the following sequences:

a) $1, 3, 5, 7, 9, 11, \ldots \implies a_n = 2n - 1$
b) $1, -4, 10, -20, 35, -56 \ldots \implies b_n = (-1)^{n+1}\frac{n(n+1)(n+2)}{6}$
c) $1, 3/2, 6, 3/24, 120, 3/720, \ldots \implies c_n = (2 + (-1)^n)n!^{(-1)^{n+1}}$
d) $1/4, -4/9, 9/16, -16/25, 25/36, \ldots \implies d_n = (-1)^{n+1}(\frac{n}{n+1})^2$
e) $1, 1/5, 1/21, 1/85, 1/341, 1/1365, \ldots \implies e_n = \frac{3}{4^n-1}$

IV. For the following sequences, Compute the first 5 elements. Then decide whether they *are* or *are not* increasing, decreasing, nonincreasing, and nondecreasing.

$$a_n = n - 3^n$$

$-2, -7, -24, -77, -238$. decreasing and nonincreasing.

$$b_n = n + \frac{1}{n}$$

$2, \frac{5}{2}, \frac{10}{3}, \frac{17}{4}, \frac{26}{5}$ increasing and nondecreasing.

$$c_n = 3 - \frac{1}{n}$$

$2, \frac{5}{2}, \frac{8}{3}, \frac{11}{4}, \frac{14}{5}$. decreasing and nonincreasing.

$$d_n = \frac{(-1)^n}{n^2}$$

$-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}$. None of them.

$$e_n = \frac{2^n + 3^n}{13n^2}$$

$\frac{5}{13}, \frac{1}{4}, \frac{35}{117}, \frac{97}{208}, \frac{275}{325}$ None of them (However is increasing and nondecreasing from the second term onwards).