HOMEWORK 6 (MATH 61, SPRING 2015)

Solve: R.J, Sec 8.1 Ex 9, 10, Sec 8.2 Ex 11, 12, 14, 15, 30, 32, 35, 36, 48, Sec 8.3 Ex 2, 4, 5, 7, 8, 14, 15, Sec 8.6 Ex 2, 3, 8, 9, 11, 12.

8.1
9. a, c, b, e, c, f, e, d, b, a
10. a, b, e, d, b, c, f, e, c, g, e, h, f, i, h, g, d, a

8.2
11. No such graph exists because the sum of the degrees of all vertices must be even.

12. 
14. No such graph exists because the sum of the degrees of all vertices must be twice the number of edges.

15. No Euler cycle because $v_3$ has odd degree
30. No Euler cycle because $v_3$ has odd degree
32. a, b, c, h, i, c, d, b, f, e, d, i, e, j, f, g, j, h, a
35. $n$ is odd
36. $m$ and $n$ are even
48. $K_2$

8.3
2. a, h, i, o, j, b, c, k, l, d, e, m, p, n, f, g, a
4. Consider two subsets of vertices, $V_1 = \{d, e, g, h, n, l\}$ and $V_2 = \{c, j, m\}$. Each vertex in $V_1$ has degree 3 and is connected to exactly 2 vertices in $V_2$. Thus, any Hamiltonian cycle must contain exactly one edge connecting each vertex in $V_1$ with a vertex in $V_2$. This implies $(a, c)$ and $(m, o)$ are not in the cycle, which in turn makes $(a, f)$ and $(f, o)$ in the cycle. But this also implies $(f, g)$ is not in the cycle which is not possible as only one of $(g, c)$ and $(g, m)$ is in the cycle.
5. Edges $(g, i), (i, j), (j, g)$ must be in any Hamiltonian cycle.
7. a, b, c, g, l, m, r, q, p, k, j, f, e, i, n, o, t, s, h, d, a
8. We may remove one of parallel edges $(a, e)$. Consider two subsets of vertices, $V_1 = \{h, d, g\}$ and $V_2 = \{c, e\}$. Each vertex in $V_1$ has degree 3 and is connected to exactly 2 vertices in $V_2$. Thus, any Hamiltonian cycle must contain at least one edge connecting each vertex in $V_1$ with a vertex in $V_2$. This is not possible as $(a, c)$ and $(a, e)$ must be in the cycle.
14. Consider a sequence of all vertices of $K_n$, $v_1, v_2, \ldots, v_n$. Then $v_1, v_2, \ldots, v_n, v_1$ is a Hamiltonian cycle.

15. $m = n \geq 2.$

8.6

2. $(a, b, c, d, e, f) \rightarrow (1, 6, 4, 5, 2, 3)$
3. $(a, b, c, d, e) \rightarrow (2, 4, 5, 1, 3)$
8. $G_1$ has 8 edges, but $G_2$ has 9 edges.
9. $G_2$ contains a subgraph isomorphic to $K_3$ ($\{1, 2, 5\}$), but $G_1$ does not.
11. $(a, b, c, d, e, f, g, h, i, j, k, l) \rightarrow (1, 5, 6, 2, 3, 7, 8, 4, 9, 10, 11, 12)$
12. $G_1$ has a vertex with degree 2 (vertex $c$), but $G_2$ does not.

**Problem I.** Draw two non-isomorphic graphs with scores (degree sequences)

a) $(3, 3, 3, 3, 5, 5, 6, 6, 6)$

[Diagram of two non-isomorphic graphs]

In the right graph, two vertices with degree 5 are connected, but not in the left graph.

b) $(3, 3, 3, 5, 5, 5, 6, 7, 7)$

[Diagram of two non-isomorphic graphs]

In the right graph, two vertices with degree 3 are connected, but not in the left graph.

**Problem II.** For each of these sequences, explain why they cannot be scores (degree sequences) of a simple graph.

a) $(2, 3, 3, 4, 5, 6, 7)$

The maximum degree of a simple graph is $|V| - 1.$
b) \((0,1,2,3,4,5,6,7)\)
A vertex with degree 7 = \(|V| - 1\) must be connected to every other vertex. Thus, every vertex must have degree at least 1.

c) \((3,3,3,5,5,5,5)\)
The sum of the degrees of all vertices must be even.

d) \((1,2,3,4,4,6,6)\)
Two vertices with degree 6 = \(|V| - 1\) must be connected to every other vertex. Thus, every vertex must have degree at least 2.

e) \((2,2,2,6,6,6,6)\)
Four vertices with degree 6 = \(|V| - 1\) must be connected to every other vertex. Thus, every vertex must have degree at least 4.

f) \((2,3,4,5,5,7,7,7)\)
Three vertices with degree 7 = \(|V| - 1\) must be connected to every other vertex. Thus, every vertex must have degree at least 3.

**Problem III.** Decide whether the following graphs are isomorphic or non-isomorphic. We include just the answers for a) and b) – the isomorphism can be found by trial and error.

a) isomorphic
b) isomorphic
c) non-isomorphic (first graph has a \(C_3\), the other does not).