Solve: RJ, Sec. 2.2 Ex 28, 29, Sec. 2.4 Ex 2, 3, 6, 9, Sec 3.2 Ex 7, 9, 13, 14.

2.2
28. Suppose that there exist positive integers \(m, n\) such that \(m^3 + 2n^2 = 36\). Then \(m^3 < 36\), and thus \(m < (36)\frac{1}{3} < 4\). Since both \(2n^2\) and 36 are even, \(m^3\) must be even, so is \(m\).

Thus, \(m = 2\). But this implies \(n^2 = 36 - 2^3 = 28\) which is not a square of any positive integer, which is a contradiction.

29. Suppose that there exist positive integers \(m, n\) such that \(2m^2 + 4n^2 - 1 = 2(m + n)\).

Then the left hand side must be odd, but the right hand side is even.

2.4
2. Base case: \(1 \cdot 2 = 2 = \frac{1 \cdot 2 \cdot 3}{3}\).

Inductive step: Suppose \(1 \cdot 2 + 2 \cdot 3 + \ldots + n(n + 1) = \frac{n(n+1)(n+2)}{3}\) for \(n \geq 1\). We want to show \(1 \cdot 2 + 2 \cdot 3 + \ldots + (n+1)((n+1) + 1) = \frac{(n+1)(n+2)(n+3)}{3}\). By the inductive assumption, we have

\[
1 \cdot 2 + 2 \cdot 3 + \ldots + (n+1)(n + 2) = \frac{n(n+1)(n+2)}{3} + (n+1)(n+2)
\]

\[
= (n+1)(n+2)\left(\frac{n}{3} + 1\right)
\]

\[
= \frac{(n+1)(n+2)(n+3)}{3}.
\]

3. Base case: \(1(1!) = 1 = 2! - 1\).

Inductive step: Suppose \(1(1!) + 2(2!) + \ldots + n(n!) = (n+1)! - 1\) for \(n \geq 1\). We want to show \(1(1!) + 2(2!) + \ldots + (n+1)((n+1)!) = (n+2)! - 1\). By the inductive assumption,

\[
1(1!) + 2(2!) + \ldots + (n+1)((n+1)!) = (n+1)! - 1 + (n+1)((n+1)!) = (n+1)! - 1 + (n+1)! = (n+2)! - 1.
\]

6. Base case: \(1^3 = 1 = \left(\frac{1 \cdot 2}{2}\right)^2\).

Inductive step: Suppose \(1^3 + 2^3 + \ldots + n^3 = \left(\frac{n(n+1)}{2}\right)^2\) for \(n \geq 1\). We want to show

\[
1^3 + 2^3 + \ldots + (n+1)^3 = \left(\frac{(n+1)(n+2)}{2}\right)^2.
\]

By the inductive assumption,

\[
1^3 + 2^3 + \ldots + (n+1)^3 = \left(\frac{n(n+1)}{2}\right)^2 + (n+1)^3
\]

\[
= (n+1)^2\left(\frac{n^2}{4} + n + 1\right)
\]

\[
= (n+1)^2\left(\frac{n^2 + 4n + 4}{4}\right)
\]

\[
= (n+1)^2\left(\frac{n + 2}{2}\right)^2 = \left(\frac{(n+1)(n+2)}{2}\right)^2.
\]

9. Base case: \(\frac{1}{2^2-1} = \frac{1}{3} = \frac{3}{4} - \frac{1}{2\cdot 1} - \frac{1}{2\cdot 3}\).

Inductive step: Suppose \(\frac{1}{2^2-1} + \frac{1}{3^2-1} + \ldots + \frac{1}{(n+1)^2-1} = \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)}\) for \(n \geq 1\).
We want to show \( \frac{1}{2^2 - 1} + \frac{1}{3^2 - 1} + \cdots + \frac{1}{(n+2)^2 - 1} = \frac{3}{4} - \frac{1}{2(n+2)} - \frac{1}{2(n+3)}. \) By the inductive assumption,

\[
\begin{align*}
\frac{1}{2^2 - 1} + \frac{1}{3^2 - 1} + \cdots + \frac{1}{(n+2)^2 - 1} &= \frac{3}{4} - \frac{1}{2(n+1)} - \frac{1}{2(n+2)} - \frac{1}{n+3} + \frac{1}{n+1} - \frac{1}{2n+3} \\
&= \frac{3}{4} - \frac{1}{2(n+2)} + \frac{1}{n+1} - \frac{1}{2n+3} \\
&= \frac{3}{4} - \frac{1}{2(n+2)} + \frac{1}{n+1} - \frac{2}{(n+1)(n+2)} \\
&= \frac{3}{4} - \frac{1}{2(n+2)} + \frac{1}{n+1} - \frac{1}{2} \\
&= \frac{3}{4} - \frac{1}{2(n+2)} + \frac{1}{n+1} \\
&= \frac{3}{4} - \frac{1}{2(n+2)}.
\end{align*}
\]

3.2

7. \( t_{2077} = 2^{(2077)} - 1 = 4153. \)

9. Note \( t_n \) is the \( n \)th odd number. The sum of the first \( n \) odd numbers is \( n^2. \) So, 13. Yes.

14. No.

I. There are \( n \) unit circles and \( n \) lines drawn in the plane. Prove that the regions in the plane separated by the lines and circles can be colored with two colors in such a way that no two adjacent regions have the same color.

The idea is very similar to the example done with lines in class. We proceed by induction. The base case is a circle and a line. There are two cases: they intersect, or not intersect, each of which can be 2-colored. Now suppose we have proved the statement for \( n \) circles and \( n \) lines.

We then add the \( (n+1) \)th circle. For every region inside the \( (n+1) \)th circle, we switch the color. Then for any pair of adjacent regions, exactly one of the following happens:

- Both regions are outside the new circle, so they had different colors before the new circle by the inductive step, and we have not altered their colors.
- Both regions are inside the circle, so they had different colors before the new circle, and we swapped both of them, so they still have different colors.
- One is outside and one is inside the new circle. Then they were part of the same region before we added the new circle, so they started out as the same color. But we swapped the color of the inside region, so now they have different colors.

Then we add the \( (n+1) \)th line. For every region on one side of the \( (n+1) \)th line, we switch the color. By similar argument, as above, every pair of adjacent regions have different colors.

II. Use induction to prove that \( 7^n - 1 \) is divisible by 6, for all \( n \geq 1. \)

Base case: \( 7 - 1 = 6 \) is divisible by 6.

Inductive step: Suppose \( 7^n - 1 \) is divisible by 6 for \( n \geq 1. \) We want to show \( 7^{n+1} - 1 \) is also divisible by 6. By the assumption, there exists an integer \( k \) such that \( 7^n - 1 = 6k. \) Then 7

\[
7^{n+1} - 1 = 7 \cdot 7^n - 1 = 7 \cdot (6k + 1) - 1 = 42k + 6 = 6(7k + 1).
\]

Since \( 7k + 1 \) is also an integer, \( 7^{n+1} - 1 \) is divisible by 6. Therefore, \( 7^n - 1 \) is divisible by 6, for all \( n \geq 1. \)
III. Find closed formulas for elements in the following sequences:
   a) 1, −3, 5, −7, 9, −11, . . . $\Rightarrow a_n = (-1)^{n+1}(2n - 1)$
   b) 1, 4, 10, 20, 35, 56, . . . $\Rightarrow b_n = \frac{n(n+1)(n+2)}{6}$
   c) 1, 1/2, 6, 1/24, 120, 1/720, . . . $\Rightarrow c_n = \frac{(n!)^{(-1)^{n+1}}}{n(2n+1)!}$
   d) 1/4, 4/9, 9/16, 16/25, 25/36, . . . $\Rightarrow d_n = \frac{(n^n)^2}{2n}$
   e) 1, 5, 21, 85, 341, 1365, . . . $\Rightarrow e_n = \frac{4^{n-1}}{3^{n-1}}$
   f) 1, 3, 15, 105, 945, 10395, . . . $\Rightarrow f_n = \frac{(2n)!}{2n^n}$

IV. For the following sequences, Compute the first 5 elements. Then decide whether they are or are not increasing, decreasing, nonincreasing, and nondecreasing.
   
   $a_n = 3^n - n$
   2, 7, 24, 77, 238. Increasing and nondecreasing.
   
   $b_n = n - \frac{1}{n}$
   0, $\frac{3}{2}$, $\frac{8}{3}$, $\frac{15}{4}$, $\frac{24}{5}$. Increasing and nondecreasing.
   
   $c_n = 3 + \frac{1}{n}$
   4, $\frac{7}{2}$, $\frac{10}{3}$, $\frac{13}{4}$, $\frac{16}{5}$. Decreasing and nonincreasing.
   
   $d_n = 3 + \frac{(-1)^n}{n^2}$
   2, $\frac{13}{4}$, $\frac{20}{9}$, $\frac{13}{4}$, $\frac{16}{5}$. None of them.
   
   $e_n = \frac{13n^2}{2n + 3n}$
   $\frac{13}{5}$, 4, $\frac{117}{35}$, $\frac{208}{97}$, $\frac{65}{57}$. None of them.