MIDTERM 1  (MATH 61, FALL 2013)

Your Name: ________________________________

UCLA id: ________________________________

Math 61 Section: __________

Date: _________________

The rules:
You MUST simplify completely and BOX all answers with an INK PEN.
You are allowed to use only this paper and pen/pencil. No calculators.
No books, no notebooks, no web access. You MUST write your name and UCLA id.
Except for the last problem, you MUST write out your logical reasoning and/or proof in full. You have exactly 50 minutes.

Points:

1
2
3
4
5

_______________

Total: (out of 100)
Problem 1. (20 points)

Compute the number of 3-subsets $A$ of $\{1, 2, \ldots, 9\}$ such that:

a) $A$ has no odd numbers,
b) $A$ has at least one number $\leq 3$,
c) the smallest number in $A$ is divisible by 3.
d) the sum of numbers in $A$ is exactly 7.
Problem 2. (20 points)

Find closed formulas for the following sequences:

a) $1, 4, 10, 31, 64, 85, 109, \ldots$

b) $2, 2, 3, 4, 6, 9, 14, 22, 35, 56, \ldots$

c) $a_1 = 1, \ a_{n+1} = a_n \cdot \binom{n+1}{2}$

d) $a_1 = 1, \ a_2 = 1, \ a_{n+1} = a_{n-1} - a_n$ for $n \geq 2$.

Note: you can express $a_n$ in terms of Fibonacci numbers $F_n$. 
Problem 3. (15 points)

Let $a_n = 1111 \cdots 1$ ($n$ ones). Suppose $a_k$ is divisible by 97. Use induction to show that $a_{k^n} = 0 \mod 97$, for all $n \geq 1$. 
Problem 4. (15 points)

There are 6 candidates in a student election, to form a 3-student committee. The total number of students is 105. Each student votes for exactly three candidates (of the 6). Prove that at least one candidate had 53 or more votes.
Problem 5. (30 points, 2 points each) TRUE or FALSE?

Circle correct answer with ink. No explanation required.

T F (1) The number of functions from \( \{ A, B, C \} \) to \( \{ 1, 2, 3, 4, 5 \} \) is equal to 15.

T F (2) The relation \( R \) on all integers \( \mathbb{Z} \) is defined by \( xRy \) if and only if \( x^2 - y^2 = 1 \). Then \( R \) is a transitive relation.

T F (3) The relation \( R \) on all integers \( \mathbb{Z} \) is defined by \( xRy \) if and only if \( x^2 + y^2 \leq 1 \). Then \( R \) is a transitive relation.

T F (4) There are 4 anagrams of the word MAMA.

T F (5) There are infinitely many Fibonacci numbers which are divisible by 3.

T F (6) The number of permutations of \( \{ 1, 2, 3, 4, 5 \} \) is smaller than 123.

T F (7) The number of 3-permutations of \( \{ 1, 2, 3, 4, 5, 6 \} \) is equal to \( \binom{6}{3} \).

T F (8) Sequence \( a_n = 10n - n^2 \), \( n \geq 1 \), is increasing.

T F (9) The number of permutations of \( \{ 1, 2, \ldots, n \} \) which have \( n \) preceding \( n - 1 \) (not necessarily immediately) is equal to \( n! / 2 \).

T F (10) For every \( A, B \subset \{ 1, 2, \ldots, 12 \} \) we have \( |A \cap B| < |A \cup B| \).

T F (11) For all \( n \geq 1 \), we have
\[
\binom{2n}{0} + \binom{2n}{2} + \binom{2n}{4} + \ldots + \binom{2n}{2n} = 2^{2n-1}.
\]

T F (12) The number of grid walks from \((0, 0)\) to \((10, 10)\) going through \((3, 7)\) is equal to \( \binom{10}{3} \).

T F (13) The number of grid walks from \((0, 0)\) to \((10, 10)\) avoiding \((10, 0)\) and \((0, 10)\) is equal to \( \frac{1}{2} \binom{20}{10} \).

T F (14) The number of anagrams of MISSISSIPPI which begin with M is greater than the number of anagrams which begin with S.

T F (15) The following parabolas are drawn in the plane:
\[ y = x^2 - n^2x - n^3, \quad n = 1, \ldots, 12. \]

Then the regions of the plane separated by these parabolas can be colored with two colors in such a way that no two adjacent regions have the same color.