MATH 61: HOMEWORK 2

3.3
25. not reflexive, not symmetric, antisymmetric, transitive, not a partial order
26. reflexive, not symmetric, antisymmetric, transitive, partial order
28. reflexive, symmetric, not antisymmetric, transitive, not a partial order (this is the same as saying \((x, y) \in R\) if \(x = y \pmod{3}\))
29. reflexive, symmetric, not antisymmetric, transitive, not a partial order (this relation is actually the same as the one in 28)

3.4
6. This is the same as saying \((x, y) \in R\) if \(x = y \pmod{4}\), and it is an equivalence relation. The equivalence classes are: \(\{1, 5\}, \{2\}, \{3\}, \{4\}\).
8. This is not even reflexive: eg. 3 does not divide \(2 - 3 = -1\), so it is not an equivalence relation.
10. This relation is not transitive: Say A and B lived in the same country, but then B moved abroad, and now lives in the same country as C. Nothing guarantees that A and C have ever lived in the same country. Hence this is not an equivalence relation.
11. This is an equivalence relation.
13. This is an equivalence relation.
14. This is an equivalence relation.

6.1
6. There are six positions, we can choose each of them to be a dot or not, this gives \(2^6\) options. But one of these are the case when no dot is present, which is not allowed. So there are \(2^6 - 1\) possible characters.
8. If repetitions are allowed: \(26 \cdot 26 \cdot 26 \cdot 10 \cdot 10\). If repetitions are not allowed: \(26 \cdot 25 \cdot 24 \cdot 10 \cdot 9\)
42. \(5 \cdot 4 + 5 \cdot 4 \cdot 3\)
43. \(2 \cdot 5 \cdot 4\)
88. 5-4 selections when Ben is chairperson, 5-4 selections when Alice is secretary, 4 selections when both of these happen. So by inclusion-exclusion, the number of selections when Ben is chairperson or Alice is secretary is \(5 \cdot 4 + 5 \cdot 4 - 4\).
90. 6 outcomes have the blue die showing 3, 6 \(\cdot 3\) outcomes have an even sum, and 3 outcomes have both. The answer is \(6 + 6 \cdot 3 - 3\).
91. \(10000/5 = 2000\) integers from 1 to 10000 are multiples of 5, \([10000/7] = 1428\) of them are multiples of 7, \([10000/35] = 285\) of them are multiples of both 5 and 7 (or equivalently multiples of 35). By inclusion-exclusion, \(2000 + 1428 - 285\) integers are multiples of at least one of the numbers 5 and 7.

6.2
6. \(11 \cdot 10 \cdot 9 \cdot 7 \cdot 6\)
8. \(12 \cdot 11 \cdot 10 \cdot 9\)
29. \(\binom{12}{4}\)
34. \(\binom{9}{1} \cdot \binom{9}{3}\)
35. We can either calculate the number of ways based on the number of women to get \(\binom{13}{4} \cdot \binom{6}{0} + \binom{13}{3} \binom{6}{1} + \binom{13}{2} \binom{6}{2} + \binom{13}{1} \binom{6}{3}\), or note that there are exactly \(\binom{13}{4}\) cases when there is no woman on the committee, so there are \(\binom{13}{4} - \binom{6}{0}\) ways to choose a committee with at least one woman.
37. There are \( \binom{n}{4} \) ways to choose a committee with no woman on it, and \( \binom{n}{1} \) ways without a man. So the answer is \( \binom{n}{1} - \binom{n}{4} - \binom{4}{1} \).

I.

a) \( a_1a_n = 2 \) means that either \( a_1 = 1 \) and \( a_n = 2 \), or \( a_1 = 2 \) and \( a_n = 1 \). In both cases we can assign the remaining \( n - 2 \) values for \( a_2, ..., a_{n-1} \) arbitrarily, so there are a total of \( 2 \cdot (n-2)! \) permutations with \( a_1a_n = 2 \). With \( n = 12 \), this is \( 2 \cdot 10! \).

b) The largest difference we can have is \( n - 1 \), so there is no way \( a_1 - a_n = n \). The answer is 0.

c) We can choose \( a_1 \) in \( n \) ways, and this uniquely determines \( a_n = n + 1 - a_1 \). (Note that this \( a_n \) will always be different than \( a_1 \), since we know \( n \) is even.) With fixed \( a_1 \) and \( a_n \), we have \( (n-2)! \) ways to assign the remaining values to \( a_2, ..., a_{n-1} \). Therefore the answer is \( n \cdot (n-2)! \), which is \( 12 \cdot 10! \) for \( n = 12 \).

d) \( (n-2)! \), or for \( n = 12 \): 10!

e) Number of permutations with \( a_1 = 1 \) is \( (n-1)! \), number of permutations with \( a_2 = 1 \) is \( (n-1)! \). Since they cannot both happen at the same time, the number of permutations when at least one of them is true is \( 2 \cdot (n-1)! \) (2 \cdot 11! for \( n = 12 \))

f) Number of permutations with \( a_1 = 1 \) is \( (n-1)! \), number of permutations with \( a_2 = 2 \) is \( (n-1)! \), number of permutations when both happen is \( (n-2)! \), so the answer is \( 2 \cdot (n-1)! - (n-2)! \) (2 \cdot 11! - 10! for \( n = 12 \))

g) Inclusion-exclusion with three sets says that \( |X \cup Y \cup Z| = |X| + |Y| + |Z| - |X \cap Y| - |X \cap Z| - |Y \cap Z| + |X \cap Y \cap Z| \). Using this we get that the number of permutations here is \( 3 \cdot (n-1)! - 3 \cdot (n-2)! + (n-3)! \) (3 \cdot 11! - 3 \cdot 10! + 9! for \( n = 12 \)).

II.

a) We need \( k - 2 \) other elements from \( \{2, ..., n-1\} \). There are \( \binom{n-2}{k-2} = \binom{n}{2} \) ways to choose them.

b) We need \( k - 1 \) elements from \( \{2, ..., n-1\} \). There are \( \binom{n-2}{k-1} = \binom{n}{3} \) ways to choose them.

c) \( \binom{n-1}{k-1} + \binom{n-1}{k-1} - \binom{n-2}{k-2} = 2 \cdot \binom{n}{3} - \binom{8}{3} \).

d) There are \( \binom{n-4}{k-2} \) choices which do not contain at least one integer \( \leq 4 \). So the answer is \( \binom{n}{k} - \binom{n-4}{k-2} = \binom{10}{4} - \binom{6}{2} \).

e) \( A \) has only four elements, so it is always missing at least one integer \( \leq 6 \) (pigeonhole principle). So we can forget about the second condition. Similarly to d), we get \( \binom{n}{k} - \binom{n-3}{k-3} = \binom{10}{4} - \binom{6}{2} \)

f) \( \binom{10}{4} - \binom{3}{1} - \binom{4}{1} \). (Or \( \binom{8}{2} \cdot \binom{8}{3} + \binom{8}{1} \cdot \binom{4}{1} + \binom{8}{1} \))

g) \( \binom{4}{1} \)