2.2.
28. Note that if \( m \geq 4 \), then \( m^3 \geq 4^3 = 64 > 36 \), so there can be no solution. Also, if \( m \) is odd, then \( m^3 \) is odd, and so \( 2n^2 = 36 - m^3 \) is odd, which is impossible. This leaves \( m = 2 \). Then \( 2n^2 = 36 - 8 = 28 \), or \( n^2 = 14 \), which is impossible since 14 is not a square.

29. If \( m \) and \( n \) are integers, the left hand side must be odd, while the right hand side must be even, so there is no solution.

2.4.
2. Base case: \( n = 1 \). \( 1 \cdot 2 = 2 = \frac{1(2)(3)}{3} \).
Induction step: suppose we have proven the equality for \( n = k - 1 \). We use this to prove the case \( n = k \).
We have
\[
1 \cdot 2 + \ldots + (k - 1)k + k(k + 1) = \frac{(k - 1)(k)(k + 1)}{3} + k(k + 1) \quad \text{by the induction hypothesis}
\]
\[
= \frac{(k - 1)(k)(k + 1) + 3k(k + 1)}{3}
\]
\[
= \frac{(k - 1)(k)(k + 1) + 3k(k + 1)}{3}
\]
\[
= \frac{(k - 1)(k)(k + 1) + 3k(k + 1)}{3}
\]
\[
= \frac{(k + 2)(k)(k + 1)}{3}
\]
\[
= \frac{k(k + 1)(k + 2)}{3}.
\]

3. Base case: \( n = 1 \). \( 1! = 1 = 2 - 1 = (1 + 1)! - 1 \).
Induction step: suppose we have proven the equality for \( n = k - 1 \). We use this to prove the case \( n = k \).
We have
\[
1! + \ldots + (k - 1)(k - 1)! + k(k!) = k! - 1 + k(k!) = (k + 1)! - 1 = (k + 1)! - 1.
\]

6. Base case: \( 1^3 = 1 = \left( \frac{1(2)}{2} \right)^2 \).
Induction step: suppose true for \( n = k - 1 \). For \( n = k \),
\[
1^3 + \ldots + (k - 1)^3 + k^3 = \left( \frac{(k - 1)k}{2} \right)^2 + k^3
\]
\[
= \frac{(k - 1)^2k^2}{4} + 4k^3
\]
\[
= \frac{((k - 1)^2 + 4k)(k^2)}{4}
\]
\[
= \frac{(k + 1)^2(k^2)}{4}
\]
\[
= \left( \frac{k(k + 1)}{2} \right)^2.
\]
9. Base case: \( \frac{1}{2^2-1} = \frac{1}{3} \), and \( \frac{3}{4} - \frac{1}{2(4)} - \frac{1}{2(3)} = \frac{1}{2} - \frac{1}{6} = \frac{1}{3} \).

Induction step: suppose true for \( n = k - 1 \). For \( n = k \),
\[
\frac{1}{2^2-1} + \ldots + \frac{1}{k^2-1} + \frac{1}{(k+1)^2-1} = \frac{3}{4} - \frac{1}{2k} - \frac{1}{2(k+1)} + \frac{1}{(k+1)^2-1} = \frac{3}{4} - \frac{k+2}{2k(k+2)} - \frac{1}{2(k+1)} + \frac{2}{2k(k+2)} = \frac{3}{4} - \frac{k}{2k(k+2)} - \frac{1}{2(k+1)} = \frac{3}{4} - \frac{1}{2k+2} - \frac{1}{2(k+1)}.
\]

3.2.

7. \( t_{2077} = 2(2077) - 1 = 4153 \).

9. Note \( t_n \) is the \( n \)th odd number. The sum of the first \( n \) odd numbers is \( n^2 \). So,
\[
\sum_{i=3}^{7} t_i = \sum_{i=1}^{7} t_i - 2 \sum_{i=1}^{2} t_i = 7^2 - 2^2 = 45.
\]

13. Yes.

14. No.

I. The idea is very similar to the example done with lines in class. We proceed by induction. The base case is a single circle. We can color the inside red and the outside blue. Now suppose we have proved the statement for \( k - 1 \) circles. When we add the \( k \)th circle, for every region inside the \( k \)th circle, we switch the color. Then for any pair of adjacent regions, exactly one of the following happens:

- Both regions are outside the new circle, so they had different colors before the new circle by the inductive step, and we have not altered their colors.
- Both regions are inside the circle, so they had different colors before the new circle, and we swapped both of them, so they still have different colors.
- One is outside and one is inside the new circle. Then they were part of the same region before we added the new circle, so they started out as the same color. But we swapped the color of the inside region, so now they have different colors.

II. I’ll start with \( n = 1 \).

a. \( 2n - 1 \).

b. \( \frac{n(n+1)}{2} \).

c. \( (-1)^{n+1} n! \).

d. \( \frac{n-1}{n} \).

e. \( 4^{n-1} \).

f. \( a_1 = 1, a_n = n + (-1)^n \) for \( n \geq 2 \).

III. a. 1, 2, 5, 12, 27. Increasing and nondecreasing.

b. 2, 2 + \( \frac{1}{2} \), 3 + \( \frac{1}{3} \), 4 + \( \frac{1}{4} \), 5 + \( \frac{1}{5} \). Increasing and nondecreasing.

c. 2, 2 + \( \frac{1}{2} \), 2 + \( \frac{2}{3} \), 2 + \( \frac{3}{4} \), 2 + \( \frac{4}{5} \). Increasing and nondecreasing.

d. 2, 3 + \( \frac{1}{2} \), 2 + \( \frac{2}{3} \), 3 + \( \frac{1}{4} \), 2 + \( \frac{3}{5} \). None of them.

\( \frac{3}{2} \), 3, \( \frac{27}{8} \), 3, \( \frac{327}{52} \). None of them.