MIDTERM 1 SOLUTIONS (4653, FALL 2008)

Problem 1. (10 points)
Assume A, B, C are mutually independent events, $P(A) = 1/2$, $P(B) = 2/3$, $P(C) = 3/5$. Compute:

a) $P(A \cap B)$
b) $P(A \cup B \mid C)$
c) $P(A \cap B \mid B \cup C)$
d) $P(\overline{A} \cup \overline{B} \cup \overline{C})$

****

$$P(A \cap B) = P(A) \cdot P(B) = \frac{1}{3}$$

$$P(A \cup B|C) = P(A \cup B) = P(A) + P(B) - P(A \cap B) = \frac{1}{2} + \frac{2}{3} - \frac{1}{3} = \frac{5}{6}$$

$$P(A \cap B \mid B \cup C) = \frac{P(A \cap B) \cap (B \cup C)}{P(B \cup C)} = \frac{P(A \cap B)}{P(B \cup C)} = \frac{1/3}{2/3 + 3/5 - 2/3 \cdot 3/5} = \frac{1/3}{13/15} = \frac{5}{13}$$

$$P(\overline{A} \cup \overline{B} \cup \overline{C}) = 1 - P(A \cap B \cap C) = 1 - \frac{1}{2} \cdot \frac{2}{3} \cdot \frac{3}{5} = 1 - \frac{1}{5} = \frac{4}{5}$$

Problem 2. (10 points)
A town has a basketball and a football team which play in tournaments once a week. Assume the basketball team wins with probability 1/3 and the football team wins with probability 2/5 and the team's performance is independent of each other.

a) Compute the expected wait time until both teams *win* on the same week.

b) Compute the expected wait time until both teams *loose* on the same week.

****

$$P(\text{both teams win}) = \frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$$

Therefore for X = number of weeks until both teams win we have $X = Geo(2/15)$ and $E[X] = 15/2 = 7.5$

$$P(\text{both teams loose}) = \frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$$

Therefore for Y = number of weeks until both teams win we have $Y = Geo(2/5)$ and $E[Y] = 5/2 = 2.5$

Problem 3. (10 points)
The urn has 12 balls, of which 4 are green and 8 are white. Two players take turns removing balls from the urn (without replacement) until no balls are left. Compute

a) Probability that the first player ends up with all 4 green balls

b) Probability that both players end up with the same number of white balls.

****

For *a*), our P = P(second ends up with no green balls) =

$$\frac{\binom{8}{6}}{\binom{12}{6}}$$

For *b*), our P = P(first ends up with 2 green balls and 4 white balls) =

$$\frac{\binom{4}{2}\cdot\binom{8}{4}}{\binom{12}{6}}$$

Problem 4. (10 points)
There are 30 students in class. A teacher tabulates the number of books students brought to class. Turns out, there are $2, 6, 6, 10, 6$ students who have $1, 2, 3, 4, 5$ books, respectively. If X is the number of books a random student has, compute $E(X)$ and $Var(X)$.

****

$$E[X] = \frac{2}{30}\cdot 1 + \frac{6}{30}\cdot 2 + \frac{6}{30}\cdot 3 + \frac{10}{30}\cdot 4 + \frac{6}{30}\cdot 5 = \frac{2}{30}(1+6+9+20+15) = \frac{51}{15} = \frac{17}{5} = 3.4$$

$$E[X^2] = \frac{2}{30}\cdot 1^2 + \frac{6}{30}\cdot 2^2 + \frac{6}{30}\cdot 3^2 + \frac{10}{30}\cdot 4^2 + \frac{6}{30}\cdot 5^2 = \frac{2}{30}(1+12+27+80+75) = \frac{195}{15} = 13$$

$Var(X) = 13 - 3.4^2 = 1.44$

Problem 5. (10 points)
Suppose there are 18 people at a party. Two people are "surprised" if they have the same Zodiac sign (i.e. their birthdays fall on the same roughly month long period). Assuming there are 12 signs in a year and the birthdays are uniformly distributed among them, compute the expected number of "surprised" pairs of people.

****

The number of pairs of people is $\binom{18}{2} = 153$. The probability that two people have the same sign is $1/12$. Thus the average number of surprised pairs is $153/12 = 12.75$.

Problem 6. (10 points)
Denote by $(a_1, \dots, a_8)$ a random permutation of $\{1, \dots, 8\}$. Compute:
a) $P(a_2 + a_4 = 4)$
b) $P(a_1 > a_2 > a_3 > a_4)$
c) $E[X]$, where X is the number of times i and a_i have the same parity, $i = 1..8$.

****

$$P(a_2 + a_4 = 4) = \frac{2 \cdot 6!}{8!}$$

$$P(a_1 > a_2 > a_3 > a_4) = P(\text{of all permutations of} a_1, a_2, a_3, a_4 \text{they non-increase}) = \frac{1}{24}$$

$$P(a_i \text{ has the same parity as } i) = \frac{4}{8} = \frac{1}{2}$$

Therefore, the average number of such i is $E[X] = 4$.

Problem 7. (10 points)
Suppose random variable X has Poisson distribution $\mathrm{Poi}(\lambda)$, such that

$$P(X=2) \;=\; 12 \cdot P(X=4).$$

a) Find λ.
b) Compute $P(X \geq 4 \mid X \geq 2)$.
c) Compute $P(X=17)$.

****

$P(X=2) = e^{-\lambda}\lambda^2/2$, $12 \cdot P(X=4) = 12e^{-\lambda}\lambda^4/24 = e^{-\lambda}\lambda^4/2$. Thus $\lambda = 1$.

$$\begin{aligned} P(X \geq 4 \mid X \geq 2) = \frac{P(X \geq 4)}{P(X \geq 2)} &= \frac{1 - P(X=0) - P(X=1) - P(X=2) - P(X=3)}{1 - P(X=0) - P(X=1)} \\ &= \frac{1 - 1/e - 1/e - 1/2e - 1/6e}{1 - 1/e - 1/e} = \frac{e - 2\frac{2}{3}}{e-2} \end{aligned}$$

$P(X=17) = e^{-1}1^{17}/17! = 1/e \cdot 17!$

Problem 8. (30 points, 3 points each). **True or False?**
Give answers only, next to the questions. No reasoning/calculations will be taken into account.

a) For every discrete random variable X, we have $E[X^2] \geq Var(X)$.

TRUE

b) In the 100 rabbits problem, the events A and B are independent, where
$A =$ first shooter kills first rabbit,
$B =$ first and second shooter kill the same rabbit.

TRUE

c) If X and Y are independent, then so are $(X + 1)$ and $(2Y - 1)$.

TRUE

d) Suppose X has $\text{Bin}(100, 0.1)$ distribution. Then X is a sum of 10 independent random variables with $\text{Ber}(0.01)$ distribution.

FALSE

e) The are two groups of students: first group has 15 freshmen and the second group has 25 seniors. Suppose the average GPA is 3.8 in each group, even though the freshmen have only 8 grades each, while the seniors have 25 grades each. Then the average GPA of all students must also be 3.8.

FALSE

f) $P(A \cup B) \geq 1 - P(A) - P(B)$ for all events A, B.

FALSE

g) Suppose random variables X and Y are independent and have $\text{Geo}(0.1)$ and $\text{Geo}(0.2)$ distributions. Then $E[X] - E[Y] = 10$.

FALSE

h) In the coupon collector's problem with 50 coupons, collecting the last coupon on average takes longer than collecting the first 20 coupons.

TRUE

i) According to the *Markov inequality*, $P(X \geq a \cdot \mu_X) \leq 1/a^2$.

FALSE

j) According to the *Bayes formula*,

$$\frac{P(A)}{P(B)} = \frac{P(A \mid B)}{P(B \mid A)}.$$

TRUE