HOME ASSIGNMENT 1 (MATH 206, FALL 2012)

1. Let $A = (a_{ij})$, where $a_{ij} = 1$ for $i + j \leq n + 2$, and $a_{ij} = 0$ otherwise $(1 \leq i, j \leq n)$. Find the permanent $\mathrm{Per}(A)$.

2. Fix $0 < b_1, \ldots, b_n < 1/2$, and let $B = \{0 \leq x_i \leq b_i, 1 \leq i \leq n\} \subset \mathbb{R}^n$ be a brick. Consider a tile $T = [0,1]^n \smallsetminus B$. Prove that $\mathbb{R}^n$ can be tiled with copies of T.

3. Denote by $c(\sigma)$ the number of cycles in $\sigma \in S_n$. Compute the sums

$$\sum_{\sigma \in S_n} c(\sigma) \qquad \text{and} \qquad \sum_{\sigma \in A_n} c(\sigma)\,,$$

where $A_n \subset S_n$ is the set of even permutations.

4. Denote by L_n the number of 3-connected plane triangulations with n vertices. Prove that L_n grows exponentially. Prove that there exists a $\lim_{n\to\infty} \frac{1}{n} \log L_n$.

5. Let t_n be a plane tree with n vertices, chosen uniformly at random. Estimate the expectation and the variance of the degree of the root vertex in t_n.

6. Prove that partition function $p(n)$ changes parity infinitely many times. (Hint: use Euler's Pentagonal Theorem)

7. Consider all $2^{\binom{n}{2}}$ orientations of edges in K_n. For an orientation O, denote by $\boldsymbol{w}(O) = (d_1 + 1, \ldots, d_n + 1)$, where d_i is the out-degree of vertex i. Prove that each such sequence is an affine combination of permutations:

$$\boldsymbol{w}(O) \, = \sum_{\sigma \in S_n} a(\sigma) \cdot \sigma\,, \quad \text{where} \quad \sum_{\sigma \in S_n} a(\sigma) \, = \, 1, \quad \text{and} \quad a(\sigma) \geq 0 \quad \text{for all} \quad \sigma \in S_n\,.$$

Prove also that unless $w \in S_n$, there is an even number of orientations O such that $w = \boldsymbol{w}(O)$.

8. Permutations σ and ω are said to have the *same pattern*, if $\sigma_i < \sigma_{i+1}$ whenever $\omega_i < \omega_{i+1}$, and vice versa, for all $1 \leq i < n$. Find a *pattern* (i.e. equivalence class) with the largest number of permutations.

9. Prove the following identity:

$$\frac{t}{1-t} + \frac{t^3}{1-t^3} + \frac{t^5}{1-t^5} + \frac{t^7}{1-t^7} + \ldots \, = \, \frac{t}{1-t} + \frac{t^3}{1-t^2} + \frac{t^6}{1-t^3} + \frac{t^{10}}{1-t^4} + \ldots$$

P.S. This is due Oct 19, 2012. See collaboration policy on the course webpage.