

## HOMEWORK 2 (MATH 184, WINTER 2016)

**Read:** MB, Sections 2.4, 4.1, 4.2.

**Solve:** Supplementary exercises

7, 8, 13, 14 in §2.10,  
1, 4, 5, 13, 14, 22 in §4.10 (in the last problem, use induction, not g.f.),  
and the following problems.

**I.** Let  $\sigma = (\sigma_1, \dots, \sigma_n)$  be a permutation. We say that element  $i$  is *progressive* if  $\sigma_i > i$ . We write  $\text{pro}(\sigma)$  for the number of progressive elements in  $\sigma$ . Prove that the number of  $\sigma \in S_n$  with  $\text{pro}(\sigma) = k$  is equal to Eulerian number  $A(n, k + 1)$ , for all  $0 \leq k < n$ .

**II.** Let  $Q(n, k) \subset \mathbb{R}^n$  be a polytope defined by the following inequalities:

$$0 \leq x_1, \dots, x_n \leq 1, \quad k - 1 \leq x_1 + \dots + x_n \leq k,$$

where  $1 \leq k \leq n$ . Prove that

$$\text{vol } Q(n, k) = \frac{A(n, k)}{n!}.$$

**III.** Let  $B(n) = S(n, 1) + \dots + S(n, n)$  be the total number of set partitions of  $[n]$ . For example,  $B(3) = 1 + 3 + 1 = 5$  is the number of set partitions of  $[3]$ :

$$\{1, 2, 3\}, \quad \{1, 2\}, \{3\}, \quad \{2, 3\}, \{1\}, \quad \{1, 3\}, \{2\}, \quad \{1\}, \{2\}, \{3\}.$$

Prove that the number of set partitions of  $[n]$  with no consecutive integers in the same block is  $B(n - 1)$ . For example, for  $n = 2$  only the last two set partitions satisfy this property, and indeed  $B(2) = 2$ .

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This Homework is due Wednesday January 27, at 2:59:59 pm. (right before class). Please read the collaboration policy on the course web page. Make sure you write your name in the beginning and your collaborators' names at the end. Remember to give a full proof.

P.S. Each item in the problems above has the same weight.