The persistent homology of data

Nina Otter

MPI MIS and UCLA

Oberseminar Geometrie
Leipzig, 17 July 2018
What is the topology of this set $X$ of points in $\mathbb{R}^2$?
Persistent homology

Motivation

**Idea:** thicken $X$ as $X_\epsilon = \bigcup_{x \in X} B(\epsilon; x)$ and study the topology of $X_\epsilon$. 

Problem: How do we choose $\epsilon$? If $\epsilon$ is too small:

- ...

If $\epsilon$ is too large:

- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
- ...
-..


Persistent homology

Motivation

Idea: thicken $X$ as $X_\epsilon = \bigcup_{x \in X} B(\epsilon; x)$ and study the topology of $X_\epsilon$.

Problem: How do we choose $\epsilon$?
Persistent homology

Motivation

**Idea:** thicken $X$ as $X_\epsilon = \bigcup_{x \in X} B(\epsilon; x)$ and study the topology of $X_\epsilon$.

**Problem:** How do we choose $\epsilon$?

If $\epsilon$ is too small:
Persistent homology

Motivation

**Idea:** thicken $X$ as $X_\epsilon = \bigcup_{x \in X} B(\epsilon; x)$ and study the topology of $X_\epsilon$.

**Problem:** How do we choose $\epsilon$?

If $\epsilon$ is too small:

If $\epsilon$ is too large:
Solution: Consider all possible values for $\varepsilon$ and obtain a nested sequence of spaces

$$X_{\varepsilon_1} \subseteq \cdots \subseteq X_{\varepsilon_n}, \text{ for } \varepsilon_1 \leq \cdots \leq \varepsilon_n.$$ 

Now study topological features of these spaces and how they evolve across the filtration.
Simplicial complexes

A *$k$-simplex* is the convex hull of $k + 1$ linearly independent points in Euclidean space, e.g.:

- 0-simplex
- 1-simplex
- 2-simplex
- 3-simplex

A $k$-simplex is completely determined by its $k + 1$ vertices. A *simplex* is a $k$-simplex for some $k$. 
A simplicial complex is built from simplices:
Homology of a simplicial complex

Given a simplicial complex $K$:

- the $p$th simplicial **homology** of $K$ with coefficients in a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $H_p(K)$
- The dimension of $H_p(K)$ is the $p$th **Betti number** of $K$, denoted by $\beta_p(K)$. 

Betti numbers give a count of:

- $p = 0$: connected components
- $p = 1$: holes
- $p = 2$: voids (2-dim. holes)
- $p$: the $p$-dim. holes

For example:

- $\beta_0 = 3$
- $\beta_1 = 1$
- $\beta_n = 0$ for all $n \geq 2$. 
Homology of a simplicial complex

Given a simplicial complex $K$:

- the $p$th simplicial **homology** of $K$ with coefficients in a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $H_p(K)$
- The dimension of $H_p(K)$ is the $p$th **Betti number** of $K$, denoted by $\beta_p(K)$.

Betti numbers give a count of:

- $p = 0$: connected components
- $p = 1$: holes
- $p = 2$: voids (2-dim. holes)
- $p$: the $p$-dim. holes
Homology of a simplicial complex

Given a simplicial complex $K$:

- the \textit{pth simplicial homology} of $K$ with coefficients in a field $\mathbb{K}$ is a $\mathbb{K}$-vector space $H_p(K)$
- The dimension of $H_p(K)$ is the \textit{pth Betti number of $K$, denoted by $\beta_p(K)$}.

Betti numbers give a count of:

- $p = 0$: connected components
- $p = 1$: holes
- $p = 2$: voids (2-dim. holes)
- $p$: the $p$-dim. holes

\[
\begin{align*}
\beta_0 &= 3 \\
\beta_1 &= 1 \\
\beta_n &= 0 \text{ for all } n \geq 2.
\end{align*}
\]
Functoriality of homology

A map of simplicial complexes $f : K \to K'$ induces a map $H_p(f) : H_p(K) \to H_p(K')$ on the homology vector spaces.

\footnote{G. Carlsson, Topology and data, Bulletin of the AMS, 2009}
A map of simplicial complexes $f : K \to K'$ induces a map $H_p(f) : H_p(K) \to H_p(K')$ on the homology vector spaces.

Given $K \xrightarrow{f} K' \xrightarrow{g} K''$ we have $H_p(g \circ f) = H_p(g) \circ H_p(f)$.

---

\(^1\)G. Carlsson, Topology and data, Bulletin of the AMS, 2009
Functoriality of homology

A map of simplicial complexes $f : K \rightarrow K'$ induces a map $H_p(f) : H_p(K) \rightarrow H_p(K')$ on the homology vector spaces.

Given $K \xrightarrow{f} K' \xrightarrow{g} K''$ we have $H_p(g \circ f) = H_p(g) \circ H_p(f)$.

---

$^1$G. Carlsson, Topology and data, Bulletin of the AMS, 2009
Functoriality of homology

A map of simplicial complexes $f : K \to K'$ induces a map $H_p(f) : H_p(K) \to H_p(K')$ on the homology vector spaces.

Given $K \xrightarrow{f} K' \xrightarrow{g} K''$ we have $H_p(g \circ f) = H_p(g) \circ H_p(f)$.

Functoriality has proven itself to be a powerful tool in the development of various parts of mathematics, such as Galois theory within algebra, the theory of Fourier series within harmonic analysis, and the application of algebraic topology to fixed point questions in topology. We argue that [..] it has a role to play in the study of point cloud data as well. \footnote{G. Carlsson, Topology and data, Bulletin of the AMS, 2009}
Persistent homology

Given $X \subset \mathbb{R}^2$ finite and $\epsilon > 0$ build a simplicial complex $K_\epsilon$:

$$\{x_0, \ldots, x_p\} \in K_\epsilon \text{ iff } \bigcap_{i=0}^{p} B(x_i; \epsilon) \neq \emptyset.$$
Persistent homology

Given $X \subset \mathbb{R}^2$ finite and $\epsilon > 0$ build a simplicial complex $K_\epsilon$:

$$\{x_0, \ldots, x_p\} \in K_\epsilon \text{ iff } \bigcap_{i=0}^{p} B(x_i; \epsilon) \neq \emptyset.$$  

Obtain $K_{\epsilon_1} \subseteq K_{\epsilon_2} \subseteq \cdots \subseteq K_{\epsilon_n} = K$ for $\epsilon_1 \leq \cdots \leq \epsilon_n$.

We call $(K, \{K_{\epsilon_i}\}_{i=1}^{n})$ a filtered simplicial complex.
Persistent homology

Given $X \subset \mathbb{R}^2$ finite and $\epsilon > 0$ build a simplicial complex $K_\epsilon$:

$$\{x_0, \ldots, x_p\} \in K_\epsilon \text{ iff } \bigcap_{i=0}^{p} B(x_i; \epsilon) \neq \emptyset.$$

Obtain $K_{\epsilon_1} \subseteq K_{\epsilon_2} \subseteq \cdots \subseteq K_{\epsilon_n} = K$ for $\epsilon_1 \leq \cdots \leq \epsilon_n$.

We call $(K, \{K_{\epsilon_i}\}_{i=1}^{n})$ a **filtered simplicial complex**.

Apply $p$th simplicial homology:

$$H_p(K_{\epsilon_1}) \xrightarrow{f_{1,2}} H_p(K_{\epsilon_2}) \xrightarrow{f_{2,3}} \cdots \xrightarrow{f_{n-1,n}} H_p(K_{\epsilon_n}).$$
Persistent homology

Given $X \subset \mathbb{R}^2$ finite and $\epsilon > 0$ build a simplicial complex $K_\epsilon$:

$$\{x_0, \ldots, x_p\} \in K_\epsilon \text{ iff } \bigcap_{i=0}^{p} B(x_i; \epsilon) \neq \emptyset.$$ 

Obtain $K_{\epsilon_1} \subseteq K_{\epsilon_2} \subseteq \cdots \subseteq K_{\epsilon_n} = K$ for $\epsilon_1 \leq \cdots \leq \epsilon_n$.

We call $(K, \{K_{\epsilon_i}\}_{i=1}^{n})$ a **filtered simplicial complex**.

Apply $p$th simplicial homology:

$$H_p(K_{\epsilon_1}) \xrightarrow{f_{1,2}} H_p(K_{\epsilon_2}) \xrightarrow{f_{2,3}} \cdots \xrightarrow{f_{n-1,n}} H_p(K_{\epsilon_n}).$$

More precisely, we obtain a tuple $(\{H_p(K_{\epsilon_i})\}_{i=1}^{n}, \{f_{i,j}\}_{i \leq j})$ such that $f_{k,j} \circ f_{i,k} = f_{i,j}$ for all $i \leq k \leq j$.

This is the $p$th **persistent homology** of $(K, \{K_{\epsilon_i}\}_{i=1}^{n})$. 
Persistence modules

In general,

- a sequence \( \{ M_i \}_{i \in \mathbb{N}} \) of \( \mathbb{K} \)-vector spaces
- a collection \( \{ f_{i,j} : M_i \rightarrow M_j \}_{i \leq j} \) of linear maps such that
  \[ f_{k,j} \circ f_{i,k} = f_{i,j} \]
  for all \( i \leq k \leq j \)

is called a persistence module.

What kind of object is this?

**Recall:** The ring \( \mathbb{K}[x] \) is \( \mathbb{N} \)-graded: \( \mathbb{K}[x] = \bigoplus_{i \in \mathbb{N}} \mathbb{K}x^i \).

An \( \mathbb{N} \)-graded module \( M \) over \( \mathbb{K}[x] \) is a module over \( \mathbb{K}[x] \) such that
\( M = \bigoplus_{i \in \mathbb{N}} M_i \) and \( x^j M_i \subset M_{i+j} \) for all \( i, j \).
Correspondence theorem

Theorem (Carlsson, Zomorodian, 2005\textsuperscript{2})

There is an isomorphism of categories between the category of persistence modules and the category of $\mathbb{N}$-graded modules over $K[x]$.

\textsuperscript{2}G. Carlsson, A. Zomorodian, Computing persistent homology, Discrete & Computational Geometry, 2005
Correspondence theorem

**Theorem (Carlsson, Zomorodian, 2005²)**

There is an isomorphism of categories between the category of persistence modules and the category of $\mathbb{N}$-graded modules over $\mathbb{K}[x]$.

\[
\left( \{M_i\}_{i \in \mathbb{N}}, \{f_{i,j} : M_i \to M_j\}_{i \leq j} \right) \mapsto \bigoplus_{i \in \mathbb{N}} M_i \text{ with action of } x^j \text{ on } M_i \text{ given by } f_{i,i+j}
\]

---

²G. Carlsson, A. Zomorodian, Computing persistent homology, Discrete & Computational Geometry, 2005
Correspondence theorem

Theorem (Carlsson, Zomorodian, 2005\(^2\))

There is an isomorphism of categories between the category of persistence modules and the category of \(\mathbb{N}\)-graded modules over \(\mathbb{K}[x]\).

\[
\left(\{M_i\}_{i \in \mathbb{N}}, \{f_{i,j} : M_i \to M_j\}_{i \leq j}\right) \leftrightarrow \bigoplus_{i \in \mathbb{N}} M_i \text{ with action of } x^j \text{ on } M_i \text{ given by } f_{i,i+j}
\]

\[
\left(\{M_i\}_{i \in \mathbb{N}}, \{x^j-i : M_i \to M_j\}_{i \leq j}\right) \leftrightarrow M = \bigoplus_{i \in \mathbb{N}} M_i \text{ graded module}
\]

\(^2\)G. Carlsson, A. Zomorodian, Computing persistent homology, Discrete & Computational Geometry, 2005
Structure theorem for f.g. graded modules over a PID

Theorem (Webb 1985\(^3\))

For any finitely generated \(\mathbb{N}\)-graded module \(M\) over \(K[x]\):

\[
M \cong \left( \bigoplus_{i=1}^{n} x^{\alpha_i} K[x] \right) \oplus \left( \bigoplus_{j=1}^{m} x^{\beta_j} K[x]/(x^{\beta_j+\gamma_j}) \right).
\]

This gives:

- \(n\) infinite intervals \([\alpha_i, \infty)\) for \(i = 1, \ldots, r\)
- \(m\) finite intervals \([\beta_j, \beta_j + \gamma_j)\) for \(j = 1, \ldots, m\).

This collection of intervals is called \textbf{barcode}, and it is a complete invariant for persistence modules.

---

\(^3\)C. Webb, \textit{Decomposition of graded modules}, Proceedings of the AMS, 1985
Examples of barcode

$\epsilon = 0$

$\epsilon = 0.6$

$\epsilon = 1.1$

$\epsilon = 1.6$

$\epsilon = 2.1$
Example of Barcode
Applications of PH

Persistent homology can be applied to, e.g.:

1. Finite metric spaces
2. Undirected weighted networks
3. Grey-scale digital images
PH to study grey-scale images

\[
G = \begin{pmatrix}
115 & 119 & 119 & 119 & 119 \\
115 & 94 & 94 & 94 & 114 \\
115 & 94 & 139 & 100 & 114 \\
115 & 94 & 99 & 99 & 114 \\
115 & 117 & 117 & 117 & 117 \\
\end{pmatrix}
\]
Pipeline for PH computation ($r = 1$)

- **Data depending on $r$ parameters**
- **$r$-filtered space**
- **Algebraic invariants**
Computation of 1-parameter PH

Computational challenges:

- Size of simplicial complex
- Computational complexity of matrix reduction algorithms
- Geometric properties of the space of barcodes
Computation of 1-parameter PH: current approaches

Computational challenges:

- Size of simplicial complex
  - Sparse simplicial complexes
  - Heuristic sparisification techniques

- Complexity of matrix reduction algorithms
  - Heuristic optimisations

- Geometric properties of the space of barcodes
  - Study geometry of space of barcodes
  - Topological feature vectors
A brief (biased) history of PH software

2000-2004 PH algorithm  
H. Edelsbrunner, D. Letscher, 
G. Carlsson and A. Zomorodian

2005 **Plex**  
V. de Silva, P. Perry, L. Kettner, A. Zomorodian

2011 **javaPlex**  
A. Tausz, M. Vejdemo-Johansson, H. Adams

2012 **Perseus**  
V. Nanda

2013 **PHAT**  
M. Kerber, J. Reininghaus, U. Bauer, H. Wagner

2014 **DIPHA**  
M. Kerber, J. Reininghaus, U. Bauer

2014 **GUDHI**  
C. Maria, J.D. Boissonnat, M. Glisse, M. Yvinec

2016 **Ripser**  
U. Bauer
A roadmap for the computation of persistent homology

Nina Otter$^{1,3}$, Mason A. Porter$^{4,1,2,*}$, Ulrike Tillmann$^{1,3}$, Peter Grindrod$^{1}$ and Heather A. Harrington$^{1}$

- Introduction of theory of PH to computational scientists
- Survey of state-of-the-art of computation of PH and its challenges
- Benchmarking of state-of-the-art libraries for the computation of PH
- Tutorial and guidelines for computation of PH with libraries based on benchmarking
Some applications of 1-parameter PH

- Classification of nanoporous material
- Horizontal evolution
Morphological classification of galaxies is performed using a naked-eye analysis. Issues:

- The morphology changes when the galaxy is observed at different wavelengths.
- The naked-eye method is not scalable.

We are trying to find a “topological signature” for the morphological classification of galaxies. The signature is extracted from the collection of images of a galaxy over the whole UBVRI visible band.  

---

4I. P. Martínez, N. Otter, É. R. Roa, Morphological galaxy classification by means of TDA, in preparation
Multiparameter persistent homology (MPH)

Motivation

1. Data often depend on several parameters, e.g.:
   - colored digital images
Motivation

1. Data often depend on several parameters, e.g.:
   - colored digital images
   - complex biological data sets (here: blood vessel growth in presence of tumor)
Multiparameter persistent homology (MPH)

Motivation

2. Outliers
MPH pipeline

data depending on $r$ parameters

$r$-filtered space

algebraic invariants

$r = 1$

$a$)

$b$)

$r \wedge 1$

$? $
MPH: theoretical challenges

- MPH was introduced in 2009 by Carlsson and Zomorodian\(^5\)

- Classification problem amounts to studying isomorphism classes of multigraded modules over \(K[x_1, \ldots, x_r]\)

- Desiderata for invariants for applications:
  - Computability
  - Stability
  - Interpretability

---

\(^5\)G. Carlsson, A. Zomorodian, The theory of multidimensional persistence, Discrete & Computational Geometry, 2009
Approaches focussing on finding invariants suitable for applications

1) Efficient algorithms to compute homology of multifiltrations:
   ▶ For one-critical multifiltrations\(^6\)
   ▶ For general multifiltrations\(^7\)
   ▶ del Campo, Otter (in progress): optimisation of general algorithm

2) Invariants of MPH modules coming from applications: Bauer and Botnan, Chachólski, Oudot

3) Restriction of 2-parameter PH to 1-parameter PH: Biasotti et al\(^8\), Lesnick and Wright\(^9\)

---

\(^6\) G. Carlsson, G. Singh, A. Zomorodian, Computing multidimensional persistence, ISAAC 2009, Lecture notes in computer science
\(^9\) M. Lesnick, M. Wright, Interactive Visualization of 2-D Persistence Modules, arxiv.org/1512.00180
We study submodules generated by elements supported on subspaces of support.

We propose invariants that distinguish between:

- **transient features**, support is a subspace of dimension zero,
- **partially persistent features**, support is a subspace of dimension $1 \leq d < r$, and
- **fully persistent features**, support is a subspace of dimension $r$.

---

$^{10}$H. Harrington, N. Otter, H. Schenck, U. Tillmann, Stratifying multiparameter persistent homology, arxiv.org/1708.07390
The stratification of multiparameter PH

We study\(^{10}\) submodules generated by elements supported on subspaces of support.

We propose invariants that distinguish between:

- **transient features**, support is a subspace of dimension zero,

- **partially persistent features**, support is a subspace of dimension \(1 \leq d < r\), and

- **fully persistent features**, support is a subspace of dimension \(r\).

\(^{10}\)H. Harrington, N. Otter, H. Schenck, U. Tillmann, Stratifying multiparameter persistent homology, arxiv.org/1708.07390
Future work and questions

- Software for MPH: Macaulay2 library that gives minimal presentation of MPH module

Talk tomorrow at Nonlinear Algebra Seminar!
Future work and questions

- Software for MPH: Macaulay2 library that gives minimal presentation of MPH module
- Investigate experimentally properties of MPH modules that come from applications
- → Talk tomorrow at Nonlinear Algebra Seminar!
Future work and questions

- Software for MPH: Macaulay2 library that gives minimal presentation of MPH module
- Investigate experimentally properties of MPH modules that come from applications
- Good examples of multifiltrations from applications?
Future work and questions

- Software for MPH: Macaulay2 library that gives minimal presentation of MPH module
- Investigate experimentally properties of MPH modules that come from applications
- Good examples of multifiltrations from applications?
- Euler characteristic for (M)PH?
Future work and questions

- Software for MPH: Macaulay2 library that gives minimal presentation of MPH module
- Investigate experimentally properties of MPH modules that come from applications
- Good examples of multifiltrations from applications?
- Euler characteristic for (M)PH? → Talk tomorrow at Nonlinear Algebra Seminar!