Stratifying multiparameter persistent homology

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Persistent homology pipeline

Data \rightarrow (1) \rightarrow Filtered space \rightarrow (2) \rightarrow Persistence module
Step (1): from data to filtered spaces

finite metric space \rightarrow \text{filtered simplicial complex}
Step (2): from filtered spaces to persistence modules

filtered simplicial complex \( K_{\epsilon_1} \subseteq \cdots \subseteq K_{\epsilon_n} = K \)
for \( \epsilon_1 \leq \cdots \leq \epsilon_n \)

\[
\begin{align*}
K_{\epsilon_1} & \subseteq \cdots \subseteq K_{\epsilon_n} = K \\
\text{for } \epsilon_1 & \leq \cdots \leq \epsilon_n \\
H_p(K_{\epsilon_1}) & \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n-1,n}} H_p(K_{\epsilon_n})
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\[
H_p(K_{\epsilon_1}) \xrightarrow{f_{1,2}} \cdots \xrightarrow{f_{n-1,n}} H_p(K_{\epsilon_n})
\]

More precisely, we obtain a tuple \( (\{H_p(K_{\epsilon_i})\}_{i=1}^n, \{f_{i,j}\}_{i \leq j}) \) such that \( f_{k,j} \circ f_{i,k} = f_{i,j} \) for all \( i \leq k \leq j \).

This is the \( p \text{th} \) \textbf{persistent homology} of \( (K, \{K_{\epsilon_i}\}_{i=1}^n) \).
Persistence modules

In general,

- a sequence \( \{ M_i \}_{i \in \mathbb{N}} \) of \( \mathbb{K} \)-vector spaces
- a collection \( \{ f_{i,j} : M_i \rightarrow M_j \}_{i \leq j} \) of linear maps such that \( f_{k,j} \circ f_{i,k} = f_{i,j} \) for all \( i \leq k \leq j \)

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What kind of object is this?

**Recall:** The ring \( \mathbb{K}[x] \) is \( \mathbb{N} \)-graded: \( \mathbb{K}[x] = \bigoplus_{i \in \mathbb{N}} \mathbb{K}x^i \).

An \( \mathbb{N} \)-graded module \( M \) over \( \mathbb{K}[x] \) is a module over \( \mathbb{K}[x] \) such that
\( M = \bigoplus_{i \in \mathbb{N}} M_i \) and \( x^j M_i \subset M_{i+j} \) for all \( i, j \).
Correspondence theorem

\[
\left( \{ M_i \}_{i \in \mathbb{N}}, \{ f_{i,j} : M_i \rightarrow M_j \}_{i \leq j} \right) \mapsto \bigoplus_{i \in \mathbb{N}} M_i \text{ with action of } x^j \text{ on } M_i
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given by \( f_{i,i+j} \)
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$$\left( \{ M_i \}_{i \in \mathbb{N}}, \{ x^{j-i} : M_i \to M_j \}_{i \leq j} \right) \leftarrow M = \bigoplus_{i \in \mathbb{N}} M_i \text{ graded module}$$
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Theorem (Carlsson, Zomorodian, 2005\(^1\))

*There is an isomorphism between the category of persistence modules and the category of \( \mathbb{N} \)-graded modules over \( \mathbb{K}[x] \).*

\(^1\)G. Carlsson, A. Zomorodian *Computing persistent homology*, Discrete & Computational Geometry, 2005
Structure theorem for f.g. graded modules over a PID

Theorem (Webb 1985\(^2\))

For any finitely generated \(\mathbb{N}\)-graded module \(M\) over \(K[x]\):

\[ M \cong \left( \bigoplus_{i=1}^{n} x^{\alpha_i} K[x] \right) \oplus \left( \bigoplus_{j=1}^{m} x^{\beta_j} K[x]/x^{\beta_j+\gamma_j} \right) . \]

This gives:

- \(n\) infinite intervals \([\alpha_i, \infty)\) for \(i = 1, \ldots, r\)
- \(m\) finite intervals \([\beta_j, \beta_j + \gamma_j)\) for \(j = 1, \ldots, m\).

This collection of intervals is called **barcode**, and it is a complete invariant for persistence modules.

Examples of barcode

\( \varepsilon = 0 \)

\( \varepsilon = 0.6 \)

\( \varepsilon = 1.1 \)

\( \varepsilon = 1.6 \)

\( \varepsilon = 2.1 \)
Example of Barcode
Applications of PH

Persistent homology can be applied to, e.g.:

1. Finite metric spaces
2. Undirected weighted networks
3. Grey-scale digital images
**PH to study grey-scale images**

\[ G = \begin{pmatrix} 
115 & 119 & 119 & 119 & 119 \\
115 & 94 & 94 & 94 & 114 \\
115 & 94 & 139 & 100 & 114 \\
115 & 94 & 99 & 99 & 114 \\
115 & 117 & 117 & 117 & 117 
\end{pmatrix} \]
A roadmap for the computation of persistent homology
N. Otter, M. Porter, U. Tillmann, P. Grindrod, H. Harrington,
EPJ Data Science 2017 6:17 (SpringerOpen)
Libraries for PH and overview of computation

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- Come along to the practical session on Monday 4 September
  at 3pm!
Multi-parameter persistent homology

Motivation

1. Data often depend on several parameters, e.g.:
   - colored digital images
Multi-parameter persistent homology

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   - colored digital images
   - complex biological data sets (here: blood vessel growth in presence of tumor)
Multi-parameter persistent homology

Motivation

2. Outliers
Multi-parameter persistent homology pipeline

1. Data depending on $r$ parameters
2. $r$-filtered space
3. $r$-parameter persistence module
Step (1): from data to multi-filtered spaces

Define the following partial order on $\mathbb{N}^r$:

$(u_1, \ldots, u_r) \preceq (v_1, \ldots, v_r)$ iff $u_i \leq v_i$ for all $i = 1, \ldots, r$.

A multi-filtered space $K$ is a tuple $(K, \{K_u\}_{u \in \mathbb{N}^r})$ with $K_u \subseteq K_v$ whenever $u \preceq v$ in $\mathbb{N}^r$ and $K = \bigcup_{u \in \mathbb{N}^r} K_u$. 

map $f : X \to \mathbb{R}^r$ digital image with color vectors of length $r$
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map $f : X \to \mathbb{R}^r$  $\longrightarrow$  $r$-filtered simplicial complex

digital image with color vectors of length $r$  $\longrightarrow$  $r$-filtered cubical complex
Step (2): from multi-filtered spaces to multi-parameter persistence modules

A $r$-parameter persistence module is a tuple $\left( \{ M_u \}_{u \in \mathbb{N}^r}, \{ \phi_{u,v} \}_{u \preceq v \in \mathbb{N}^r} \right)$ where:

- for each $u \in \mathbb{N}^r$ we have that $M_u$ is a $K$-vector space
- for every $u \preceq v$ we have that $\phi_{u,v} : M_u \to M_v$ is a $K$-linear map such that whenever $u \preceq u' \preceq u''$ we have

$$\phi_{u',u''} \circ \phi_{u,u'} = \phi_{u,u''}.$$

In other words, an $r$-parameter persistence module is a functor $F : \mathbb{N}^r \to K\text{Vect}.$
Correspondence theorem

What kind of objects are multiparameter persistence modules?
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Put an $\mathbb{N}^r$-grading on the ring $K[x_1, \ldots, x_r]$:

$$K[x_1, \ldots, x_r] = \bigoplus_{u \in \mathbb{N}^r} K x^u,$$

where $x^u = x_1^{u_1} \ldots x_r^{u_r}$, so every variable $x_i$ has degree $e_i \in \mathbb{N}^r$. 

Theorem (Carlsson, Zomorodian, 2009)

There is an isomorphism of categories between the category of $r$-parameter persistence modules and the category of $\mathbb{N}^r$-graded modules over $K[x_1, \ldots, x_r]$.

Problem: There is no decomposition analogous to the 1-parameter case.
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Any persistence module is the homology of a filtered space

The homology of a multifiltered space is an $\mathbb{N}^r$-graded module over $K[x_1, \ldots, x_r]$. 

[Theorem (Carlsson, Zomorodian, 2009)]

For any finitely generated $\mathbb{N}^r$-graded module $M$ there exists a finite multifiltered simplicial complex $K$ and a positive natural number $i$ such that $M$ is the homology in degree $i$ of $K$.

Therefore, studying the homology of $r$-filtered spaces amounts to studying graded modules over $K[x_1, \ldots, x_r]$. 
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Therefore, studying the homology of $r$-filtered spaces amounts to studying graded modules over $\mathbb{K}[x_1, \ldots, x_r]$. 
Desiderata for invariants for applications

Such invariants should be:

▶ Computable
▶ Stable
▶ Interpretable

From (Carlsson, Zomorodian 2009):

Our study of multigraded objects shows that no complete discrete invariant exists for multidimensional persistence. We still desire a discriminating invariant that captures persistent information, that is, homology classes with large persistence.
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Support shape of a module

**Recall:** Let $M$ be a module over a commutative ring $R$. Let $U$ be a non-empty subset of $M$. Define the **annihilator of $U$** as follows:

$$\text{Ann}(U) = \{ r \in R \mid \forall u \in U : ru = 0 \}.$$
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Denote by $\text{Ass}(M)$ the poset of all primes associated to $M$.

**Fact:** For a finitely generated $\mathbb{N}^r$-graded $K[x_1, \ldots, x_r]$-module $M$, any associated prime $p$ of $M$ is of the form $p = (x_{i_1}, \ldots, x_{i_k})$.

**Definition** For such a prime $p$ define the **complement support** $c_p = \{ (u_1, \ldots, u_r) \in \mathbb{N}^r \mid u_i = 0 \text{ for all } i \in \{i_1, \ldots, i_k\} \}$.

The **support shape** of $M$ is $\text{ss}(M) = \bigcup_{p \in \text{Ass}(M)} c_p$. 

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Stratification of the support shape

Given a sequence $p_0 \subset \cdots \subset p_m$ of associated primes of $M$, one obtains a nested sequence

$$c_{p_m} \subset \cdots c_{p_0} \subset ss(M).$$
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Definition

We call the poset \( \text{Ass}(M) \) the **stratification** of \( \text{ss}(M) \).

Note: \( \text{ss}(M) \) is completely determined by the minimal associated primes.
Example

\[ M = S(-1, -1) \oplus S(-2, -2) \]
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\[ N = S(-2, -2) \oplus S(-2, -2) \oplus \frac{S(-2, -1)}{x_2} \oplus \frac{S(-1, -2)}{x_1} \oplus \frac{S(-1, -1)}{\langle x_1, x_2 \rangle} \]
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Ass\( (M) = \{(0)\}, \) and Ass\( (N) = \{(0), (x_1), (x_2), (x_1, x_2)\}. \)
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The points of \( \mathbb{N}^r \) at which the module \( M \), as well as \( N \), does not vanish:
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The support shape of \( M \), as well as \( N \):

![Diagram of the support shape of M and N]

```plaintext
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Example (cont.)

The chains in the stratification of \( ss(N) \):
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$c(x_1, x_2) \subset c(x_2) \subset c(0)$
Generalisation of definition of birth and death

We say that a homogeneous element \( a \) of \( M \) is:

- **born at** \((u_1, \ldots, u_r) \in \mathbb{N}^r\) if the degree of \( a \) is \((u_1, \ldots, u_r)\) and \( a \) is not in the image of any sum of maps \( \sum_v x^v \) for any \( v < u \).

- If \( \text{Ann}(a) \neq (0) \) let \( D \subset \mathbb{N}^r \) be the subset of \( \mathbb{N}^r \) obtained from the set of degrees of the set of minimal generators of \( \text{Ann}(a) \) by adding to each degree the degree of \( a \). Then we say that \( a \) **dies in degrees** \( D \).

- If \( \text{Ann}(a) = (0) \) we say that \( a \) **lives forever**.

- If \( \sqrt{\text{Ann}(a)} = \langle x_{i_1}, \ldots, x_{i_k} \rangle = p \), we say that \( a \) **lives along** \( c_p \subset \mathbb{N}^r \). In this case we say that \( a \) has **support dimension** \( r - k \).

We call elements of support dimension 0 **transient components**, elements of support dimension \( 1 \leq d < r \) **persistent components**, and elements of support dimension \( r \) **fully persistent components**.
Generalisation of definition of birth and death

We say that a homogeneous element $a$ of $M$ is:

- **born at** $(u_1, \ldots, u_r) \in \mathbb{N}^r$ if the degree of $a$ is $(u_1, \ldots, u_r)$ and $a$ is not in the image of any sum of maps $\sum_v x^v$ for any $v \prec u$.

- If $\text{Ann}(a) \neq (0)$ let $D \subset \mathbb{N}^r$ be the subset of $\mathbb{N}^r$ obtained from the set of degrees of the set of minimal generators of $\text{Ann}(a)$ by adding to each degree the degree of $a$. Then we say that $a$ **dies in degrees** $D$.

- If $\text{Ann}(a) = (0)$ we say that $a$ **lives forever**.

- If $\sqrt{\text{Ann}(a)} = \langle x_{i_1}, \ldots, x_{i_k} \rangle = \mathfrak{p}$, we say that $a$ **lives along** $c_p \subset \mathbb{N}^r$. In this case we say that $a$ has **support dimension** $r - k$.

We call elements of support dimension 0 **transient components**, elements of support dimension $1 \leq d < r$ **persistent components**, and elements of support dimension $r$ **fully persistent components**.
Information forgotten by $c_p$ and $ss(M)$

Module points at which the module does not vanish

\[ M_1 = \frac{S(-1, -2)}{x_1} \]

\[ M_2 = \frac{S}{x_1^2} \]

\[ M_3 = \frac{S}{x_1} \oplus \frac{S}{x_1} \]

“translation”

“thickness”

“multiplicity”
The rank of a finitely generated module $M$ over an integral domain $R$ with field of fractions $F$ is

$$\text{rk}_S M = \dim_F M \otimes_R F$$
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The rank of a module is the minimum number of generators for the submodule generated by the fully persistent components.
The rank of a finitely generated module $M$ over an integral domain $R$ with field of fractions $F$ is

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The rank of a module is the minimum number of generators for the submodule generated by the fully persistent components.

The 0th local cohomology of $M$ along a prime $p$ is the set

$$H^0_p(M) = \{ a \in M \mid p^n a = 0 \text{ for all } n \gg 0 \}.$$
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The 0th local cohomology of $M$ along a prime $p$ is the set

$$H_p^0(M) = \{a \in M \mid p^n a = 0 \text{ for all } n \gg 0\}.$$  

If $p = \langle x_1, \ldots, x_k \rangle$ is an associated prime of $M$, then

$$H_p^0(M) = \{a \in M \mid a \text{ lives along } c_p\},$$  

and
Rank and local cohomology

- The rank of a finitely generated module $M$ over an integral domain $R$ with field of fractions $F$ is
  \[
  \text{rk}_S M = \dim_F M \otimes_R F
  \]

- The rank of a module is the minimum number of generators for the submodule generated by the fully persistent components.

- The 0th local cohomology of $M$ along a prime $p$ is the set
  \[
  H^0_p(M) = \{ a \in M \mid p^n a = 0 \text{ for all } n \gg 0 \}.
  \]
  If $p = \langle x_1, \ldots, x_k \rangle$ is an associated prime of $M$, then
  - $H^0_p(M) = \{ a \in M \mid a \text{ lives along } c_p \}$, and
  - $H^0_p(M)$ is finitely generated as an $\mathbb{K}[x_{k+1}, \ldots, x_r]$-module.
Example

module points at which the module does not vanish

\[ M_1 = \frac{S(-1,-2)}{x_1} \]

\[ M_2 = \frac{S}{x_1^2} \]

\[ M_3 = \frac{S}{x_1} \oplus \frac{S}{x_1} \]

\[ \text{rk}_{K[x_2]} H^0_{x_1}(M_1) = 1 \]

\[ \text{rk}_{K[x_2]} H^0_{x_1}(M_2) = 2 \]

\[ \text{rk}_{K[x_2]} H^0_{x_1}(M_3) = 2 \]
Conclusions

- The stratification of the support shape gives a summary of the existence of elements that live along certain coordinate directions.
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- The stability of the stratification probably depends on the codimension of the associated primes.
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▶ The stratification of the support shape gives a summary of the existence of elements that live along certain coordinate directions.

▶ The stability of the stratification probably depends on the codimension of the associated primes.

▶ Associated primes do not give information about non-trivial second syzygies (nor higher ones).