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Euler systems in the Iwasawa theory of ordinary modular forms

To Gerhard Frey, with gratitude

By

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Abstract

We formulate conjectures linking Selmer structures for Hida-theoretic and Iwasawa-theoretic families of ordinary eigenforms to Euler systems built from generalized Heegner points on towers of modular curves. We prove part of these conjectures using the method of Euler systems, control theorems and descent for Selmer complexes.

§ 1. Introduction

Let p be an odd prime and E be an elliptic curve over $\mathbb{Q}$ of conductor N with good ordinary reduction at p. Let K be a quadratic imaginary number field such that all primes dividing N split in K and let K_∞ be the $\mathbb{Z}_p$-extension of K such that $\mathrm{Gal}(K_\infty/\mathbb{Q})$ is a pro-dihedral group. The complex L-function $L(E,s)$ of E being equal to the complex L-function of a modular form by [15], it satisfies the functional equation:

$$L(E/K,s) = \epsilon(E/K,s)L(E/K,2-s)$$

The requirement on the behavior of primes dividing N forces $\epsilon(E/K,1)$ to be -1, and thus $L(E/K,1)$ to vanish at an odd order r. When r is exactly equal to 1, the general formalism of the Tamagawa Number Conjecture predicts that this vanishing is accounted for by a point z of infinite order on $E(K)$ and that the special value of $L'(E/K,s)$ at 1 is linked to the height of z, and it is indeed known by [5, 10] that the

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projection of Heegner points from the modular curve $X_0(N)$ to E give rises to such a point. Even when r is greater than 1, it was conjectured by Mazur and proved by C.Cornut and V.Vatsal that the generic order of vanishing of $L(E,\chi,s)$ at 1 is equal to 1, where $L(E,\chi,s)$ is the L-function of the Rankin-Selberg product of E with a finite order character of $\mathrm{Gal}(K_\infty/K)$. The Equivariant Tamagawa Number Conjecture (or ETNC for short) then predicts that the determinant of a well-chosen cohomology complex with coefficients in the Iwasawa algebra $\Lambda_a=\mathbb{Z}_p[[\mathrm{Gal}(K_\infty/K)]]$ should be trivialized by the p-adic height of a generic Heegner point. The Iwasawa Main Conjecture in this context, which is a p-adic variant of the Birch and Swinnerton-Dyer conjecture, then states that this trivialization should coincide with the Rankin-Selberg p-adic L-function $L_p(E)\in\Lambda$.

Using again the modularity of E and appealing to Hida theory, we see that the G_K-representation T_pE also belongs to a p-adic family of modular Galois representations parametrized by weights, or equivalently that T_pE occurs as a specialization of the inverse limit on s of the ordinary part $e^{ord}H^1_{et}(X_1(Np^s),\mathbb{Z}_p)(1)$ of the étale cohmology of the tower of modular curves $X_1(Np^s)$. The ETNC then predicts again that there exists a p-adic L-function $L_p^{Hi}(k)$ interpolating the value at $s=k/2$ of $L(f_k,s)$, that L_p^{Hi} is an element of the ordinary Hecke algebra $\mathbf{T}_\infty^{\mathrm{ord}}$ and that it induces a trivialization of the determinant of a suitably chosen cohomology complex coinciding with the trivialization given by the p-adic height of a family of Heegner points.

One drawback of the ETNC is that it itself requires several conjectures even to be formulated, for instance the existence of a motivic cohomology theory with good properties and the non-degeneracy of certain height pairings. The aim of this article is to build a conceptual framework allowing for a study of these questions independently of any conjectures. In the first section, we thus start by recalling necessary facts, though our presentation and the proofs provided are sometimes slightly non standard. Then, we state variants of the ETNC involving Euler systems, examine their compatibilities by specialization in theorem 2.8 and prove part of them in theorem 2.9. In the last section we mention generalizations to totally real fields F and to Galois representations coming from automorphic representation of the adelic point of the multiplicative group of an indefinite quaternion algebra over F.

Much of the work presented in the first section is expository in nature, and is due mostly to H.Hida and B.Howard. Theorem 2.8 is a consequence of [4], which is a joint work with T.Ochiai. The author would like to dedicate this article to G.Frey, as a small token of gratitude for the wonderful help he received from him during several critical stages of his career.

§ 2. The ETNC for p-adic families of modular G_K-representations

§ 2.1. Galois representations attached to ordinary eigenforms

General notations Let $\mathbb{A}_f$ be the finite adeles of $\mathbb{Q}$. Fix $p \geq 5$ a prime and embeddings of $\bar{\mathbb{Q}}$ into $\bar{\mathbb{Q}}_p$ and of $\bar{\mathbb{Q}}_p$ into $\mathbb{C}$. For F a finite extension of $\mathbb{Q}$ or $\mathbb{Q}_\ell$, let G_F be the absolute Galois group $\mathrm{Gal}(\bar{F}/F)$. The symbol Fr denotes geometric Frobenius morphism.

Let N be a positive integer prime to p and $k \geq 2$ an integer. The ring of diamond operators of level s is the ring $\mathbb{Z}_p[(\mathbb{Z}/p^s\mathbb{Z})^\times]$. Let $\mathbf{T}_k(N,s)$ be the image of the $\mathbb{Z}_p$-algebra generated by all Hecke operators $T(\ell)$ with $\ell \nmid N$ and all diamond operators $<a>$ of level s inside the endomorphism ring of the cuspforms of weight k for the congruence subgroup $\Gamma_0(N) \cap \Gamma_1(p^s)$. Let ϕ be a Dirichlet character of $(\mathbb{Z}/p\mathbb{Z})^\times$. Let $f \in S_k(\Gamma_0(Np), \phi)$ be an eigencuspform for the congruence group $\Gamma_0(N) \cap \Gamma_1(p)$ and let λ_f be the map such that for all $T(\ell) \in \mathbf{T}_k(N,s)$ and all $<a>$, $\lambda_f(T(\ell)) = a_\ell(f)$ and $\lambda_f(<a>) = \phi(a)$. Let $L_\mathfrak{p}$ be a fixed finite extension of $\mathbb{Q}_p$ containing the image of λ_f, let $\mathcal{O}$ be its ring of integers and $\mathbb{F}$ the residue field of $\mathcal{O}$. Let $\mathbf{T}_k(N,s,\mathcal{O})$ be $\mathbf{T}_k(N,s) \otimes_{\mathbb{Z}_p} \mathcal{O}$. We assume that f is new outside p and that it is ordinary at p, *i.e* that $a_p(f)$ belongs to $\mathcal{O}^\times$.

Let K be a quadratic imaginary extension of $\mathbb{Q}$ in which all primes dividing N split, let Σ be a finite set of finite places containing all places above Np and let $G_{K,\Sigma}$ be the Galois group of the maximal extension of K unramified outside Σ.

For G equal to a quotient of $G_{K,\Sigma}$ or G_{K_v} and for M a p-adic representation of G, let $C^\bullet_{\mathrm{cont}}(G,M)$ be the complex of continuous cochains with values in M. Whenever a complex appears in this article, it is a bounded complex whether or not this is explicitly mentioned.

Galois representations of eigenforms To f is attached an odd semi-simple residual $G_\mathbb{Q}$-representation $(\overline{T}(f), \bar{\rho}_f)_\mathbb{F}$ and an absolutely irreducible p-adic $G_\mathbb{Q}$-representation $(V(f), \rho_f)_{L_\mathfrak{p}}$. The $G_\mathbb{Q}$-representation $V(f)$ is unramified outside Np. For $\ell \nmid Np$, the trace and determinant of the geometric Frobenius $\mathrm{Fr}(\ell)$ are equal respectively to $a_\ell(f)$ and to $\phi(\ell)\chi_{cyc}^{k-1}(\ell)$, or equivalently to $\lambda_f(T(\ell))$ and $\lambda_f(<\ell>)$. We assume $\bar{\rho}_f$ to be irreducible, hence absolutely irreducible. Then there is a unique p-adic representation $(T(f), \rho_f)_\mathcal{O}$ with the same trace as $V(f)$. We note that $T(f)$ can also be considered as a free rank 2 module over $\mathbf{T}_k(N,s)$. In fact, we assume the stronger condition on f that it is not residually CM, that is to say that $\bar{\rho}_f$ restricted to G_K is absolutely irreducible.

As $V(f)$ is of dimension 2 over $L_\mathfrak{p}$, its dual $V(f)^*$ is isomorphic to $V(f)(k-1) \otimes \phi$. The form f being non-trivial, the characters χ_{cyc}^{k-2} and ϕ have the same parity so the character $\chi_{cyc}^{2-k}\phi$ factors through a group with no 2-torsion. Fix a character ψ such that $\psi^2=\chi_{cyc}^{k-2}\phi^{-1}$. We let $(V,\rho)_{L_{\mathfrak{p}}}$, $(T,\rho)_{\mathcal{O}}$ and $(\overline{T},\bar{\rho})_{\mathbb{F}}$ be respectively the self-dual $G_{\mathbb{Q}}$-representation obtained by twisting $V(f)(1)$, $T(f)(1)$ and $\overline{T}(f)(1)$ by ψ.

Let v be a place of $\mathcal{O}_K$ above p. The fact that a_p belongs to $\mathcal{O}^{\times}$ implies that the local G_{K_v}-representation T fits in a short exact sequence of non-trivial $\mathcal{O}[G_{K_v}]$-modules:

$$0\longrightarrow T_v^+\longrightarrow T\longrightarrow T_v^-\longrightarrow 0$$

We assume that $\bar{\rho}_f$ is p-distinguished, which means that $\bar{\rho}_f$ restricted to G_{K_v} for $v|p$ is not scalar.

Review of Hida theory Let Γ be the torsion-free part of $\varprojlim_s (\mathbb{Z}/p^s\mathbb{Z})^{\times}$ and Λ be the regular local ring $\mathcal{O}[[\Gamma]]$, which is also the torsion-free part of the inverse limit on s of the ring of diamond operators. Let γ be a topological generator of Γ. We allow ourselves to consider γ as an element of $G_{\mathbb{Q}}$ using the fact that Γ is isomorphic to the Galois group of the $\mathbb{Z}_p$-extension of $\mathbb{Q}$. For $k\geq 2$ and integer and ϵ a finite order character of Γ, an arithmetic point of weight k and character ϵ of Λ is an $\mathcal{O}$-algebra morphism:

$$\begin{aligned}\phi:\ \Lambda&\longrightarrow\bar{\mathbb{Q}}_p^{\times}\\ \gamma&\longmapsto\epsilon(\gamma)\chi_{cyc}^{k-2}(\gamma)\end{aligned}$$

Here, γ is considered as an element of $G_{\mathbb{Q}}$ via the identification of Γ with the Galois group of the unique $\mathbb{Z}_p$-extension of $\mathbb{Q}$. For a finite Λ-algebra R, an arithmetic point of R is an $\mathcal{O}$-algebra morphism whose restriction to Λ coincides with an arithmetic point of Λ and an arithmetic prime is the kernel of an arithmetic point.

The ring $\mathbf{T}_k(N,s,\mathcal{O})$ is a finite, flat and reduced $\mathcal{O}$-algebra and an $\mathcal{O}[(\mathbb{Z}/p^s\mathbb{Z})^{\times}]$-algebra by the inclusion of the diamond operators. Let e^{ord} be Hida's projector, this is to say the idempotent:

$$e^{ord}=\lim_{n\to\infty}T(p)^{n!}$$

The Hida ordinary Hecke algebra $\mathbf{T}_{\infty}^{\mathrm{ord}}(N,\mathcal{O})$ is the inverse limit on s of $e^{ord}\mathbf{T}_k(N,s,\mathcal{O})$. It is a torsion-free Λ-algebra independent of k, finite as Λ-module. The $\mathcal{O}$-algebra morphism λ_f is an arithmetic point of $\mathbf{T}_{\infty}^{\mathrm{ord}}(N,\mathcal{O})$ and conversely, arithmetic points of weight k of $\mathbf{T}_{\infty}^{\mathrm{ord}}(N,\mathcal{O})$ are attached to ordinary eigenforms in $S_k(\Gamma_0(N)\cap\Gamma_1(p^s))$.

Let $\mathcal{P}_{\min}$ be a minimal prime of $\mathbf{T}_{\infty}^{\mathrm{ord}}(N,\mathcal{O})$ and let $R_{\min}$ be $\mathbf{T}_{\infty}^{\mathrm{ord}}(N,\mathcal{O})/\mathcal{P}_{\min}$. As arithmetic primes of fixed weight containing $\mathcal{P}_{\min}$ are Zariski-dense in $\operatorname{Spec} R_{\min}$, the patching argument of [14] lemma 2.2.3 shows that there is a continuous $G_{\mathbb{Q}}$-pseudo-representation $\tilde{\rho}_{\min}$ with values in $R_{\min}$ interpolating the traces of the $G_{\mathbb{Q}}$-representations attached to eigencuspforms g such that $\ker\lambda_g$ contains $\mathcal{P}_{\min}$. Let $\mathfrak{m}$ be the maximal ideal of $\mathbf{T}_{\infty}^{\mathrm{ord}}(N,\mathcal{O})$ and $\mathfrak{a}$ the minimal prime contained in $\mathfrak{m}$ such that λ_f factors through the local domain $R = \mathbf{T}_\infty^{\mathrm{ord}}(N,\mathcal{O})_{\mathfrak{m}}/\mathfrak{a}$. Patching again the pseudo-representations $\tilde{\rho}_{\min}$ for all $\mathcal{P}_{\min}$ contained in $\mathfrak{m}$ defines a pseudo-representation $\tilde{\rho}_{\mathfrak{m}}$ with values in R. As $\bar{\rho}_f$ is absolutely irreducible, [13] theorem 1 shows that there is a $G_{\mathbb{Q}}$-representation $(\mathcal{T}(f),\rho_{\mathfrak{m}})_R$ free of rank 2 over R such that $\mathrm{Tr}(\rho_{\mathfrak{m}})$ is equal to $\tilde{\rho}_{\mathfrak{m}}$. The $G_{\mathbb{Q}}$-representation $\mathcal{T}(f)$ is unramified outside Np and for $\ell\nmid Np$, the characteristic polynomial of $\mathrm{Fr}(\ell)$ acting on $\mathcal{T}(f)$ is the same as the characteristic polynomial of $\mathrm{Fr}(\ell)$ acting on $T(f)$ with λ_f replaced by localization at $\mathfrak{m}$ and reduction modulo $\mathfrak{a}$.

In particular, the determinant of $\mathcal{T}(f)$ evaluated at $\mathrm{Fr}(\ell)$ for $\ell\nmid Np$ is equal to $\lambda_{\mathfrak{m}}(<\ell>)\chi_{cyc}(\ell)$. Observing that the obstruction of $\lambda_{\mathfrak{m}}(<\cdot>)$ to be a square depends only on the tame part of this character and that the tame part is the same for all specializations of $\mathcal{T}(f)$, we see that $\mathcal{T}(f)$ admits a self-dual twist $\mathcal{T}$ interpolating the T exactly as $\mathcal{T}(f)$ interpolates the $T(f)$. Let v be a place of $\mathcal{O}_K$ above p. The local G_{K_v}-representation $\mathcal{T}$ fits in a short exact sequence of non-trivial $\mathcal{O}[G_{K_v}]$-modules:

$$0\longrightarrow \mathcal{T}_v^{+}\longrightarrow \mathcal{T}\longrightarrow \mathcal{T}_v^{-}\longrightarrow 0$$

Geometric realization of $\mathcal{T}(f)$ and control theorem for $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$ We review important and well-known commutative algebra properties of $\mathcal{T}(f)$ and $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$.

Assume first that f is of weight 2, which is a very mild assumption to make as $\mathcal{T}(f)$ certainly has plenty of specializations of weight 2. Then, [15, Theorem 2.1] states that there are isomorphisms of Hecke and $G_{\mathbb{Q}}$-modules

$$\begin{aligned}\overline{T}(f) &\overset{\sim}{\longrightarrow} e_{\mathfrak{m}}^{ord}H^1_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})[\mathfrak{m}]\\ T(f) &\overset{\sim}{\longrightarrow} e_{\mathfrak{m}}^{ord}H^1_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})\end{aligned}$$

and that $\mathbf{T}_2^{\mathrm{ord}}(N,s)_{\mathfrak{m}}$ is a Gorenstein ring. This in turn implies the following control theorem.

Proposition 2.1. *Let λ be an arithmetic point of $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$ of weight k and level s and let $\mathfrak{p}$ be the kernel of λ restricted to Λ. Then $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}\otimes_\Lambda\Lambda/\mathfrak{p}$ is isomorphic to $\mathbf{T}_k^{\mathrm{ord}}(N,s)_{\mathfrak{m}}$.*

Proof. Assume first that $k=2$. For $s\geq 1$, let $C(s)$ be the complex of $\mathbf{T}_2^{\mathrm{ord}}(N,1)_{\mathfrak{m}}$-modules $e_{\mathfrak{m}}^{ord}\,\mathrm{R}\,\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})$ and let $M(s)$ be its first cohomology group. Let $C(\infty)$ be the complex of $\mathbf{T}_{\infty,\mathfrak{m}}^{\mathrm{ord}}$-modules

$$\varprojlim_s e_{\mathfrak{m}}^{ord}\,\mathrm{R}\,\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})$$

and let $M(\infty)$ be its first cohomology group. The $\mathbf{T}_2^{\mathrm{ord}}(N,1)_{\mathfrak{m}}$-module $M(1)$ is free and is the only non-trivial cohomology group of $C(1)$, which is thus a perfect complex. The isomorphism

$$\mathrm{R}\,\Gamma_{et}(X_1(Np^{s'})\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})\overset{\mathrm{L}}{\otimes}_{\mathcal{O}[(\mathbb{Z}/p^{s'}\mathbb{Z})]}\mathcal{O}[\mathbb{Z}/p^s\mathbb{Z}]\overset{\sim}{\longrightarrow}\mathrm{R}\,\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O}) \tag{2.1}$$

shows that $C(\infty)$ is a perfect complex, that $M(\infty)$ is Λ-free and $\mathbf{T}^{\mathrm{ord}}_{\infty,\mathfrak{m}}$-free of rank 2 and that $M(\infty)\otimes_\Lambda\Lambda/\mathfrak{p}$ is isomorphic to $M(s)$. Hence, $\mathbf{T}^{\mathrm{ord}}_{\infty,\mathfrak{m}}$ is a Gorenstein ring which is a Λ-free module and $\mathbf{T}^{\mathrm{ord}}_{\infty,\mathfrak{m}}\otimes_\Lambda\Lambda/\mathfrak{p}$ is isomorphic to $\mathbf{T}^{\mathrm{ord}}_k(N,s)_{\mathfrak{m}}$.

Now $\mathfrak{p}$ is of arbitrary weight. Let $\mathcal{F}$ be the usual coefficient sheaf of weight $k-2$ used to realize $G_{\mathbb{Q}}$-representations of modular forms. The ordinary étale cohomology complex satisfies the independence of weight property:

$$\varprojlim_s e^{ord}_{\mathfrak{m}}\,\mathrm{R}\,\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})\overset{\sim}{\longrightarrow}\varprojlim_s e^{ord}_{\mathfrak{m}}\,\mathrm{R}\,\Gamma_{et}(X_1(Np^s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{F})$$

The isomorphism (2.1) with coefficient sheaf $\mathcal{F}$ and the fact that

$$e^{ord}_{\mathfrak{m}}H^1_{et}(X_1(Np^s),\mathcal{F}\otimes_{\mathcal{O}}L_{\mathfrak{p}})$$

is $\mathbf{T}^{\mathrm{ord}}_k(N,s)\otimes_{\mathcal{O}}L_{\mathfrak{p}}$-free of rank 2 show that the $\Lambda/\mathfrak{p}$-module $\mathbf{T}^{\mathrm{ord}}_\infty\otimes_\Lambda\Lambda/\mathfrak{p}$ is free of rank 2, that it surjects onto $\mathbf{T}^{\mathrm{ord}}_k(N,s)$ and that this surjection becomes an isomorphism after inverting p. Hence, the $\Lambda/\mathfrak{p}$-modules $\mathbf{T}^{\mathrm{ord}}_\infty\otimes_\Lambda\Lambda/\mathfrak{p}$ and $\mathbf{T}^{\mathrm{ord}}_k(N,s)$ are isomorphic. □

Remark: We seize this opportunity to alert readers of a mistake in the literature originating in [12, Theorem 7]: contrary to what is claimed in this theorem and in subsequent quotations, the Hecke algebra localized at an ordinary, non-Eisenstein, maximal ideal is not necessarily a Gorenstein ring (see the many counter-examples found by G.Wiese and L.Kilford). In other words, our hypothesis that $\bar{\rho}_f$ is p-distinguished is necessary.

Review of dihedral Iwasawa theory Let K_∞ be the unique $\mathbb{Z}_p$-extension of K such that $\mathrm{Gal}(K_\infty/K)$ is a pro-dihedral group and let K_n be the sub-extension of K with Galois group $\mathbb{Z}/p^n\mathbb{Z}$. We fix a character χ_a of G_K with values in $\mathbb{Z}_p^\times$ identifying $\mathrm{Gal}(K_\infty/K)$ with $1+p\mathbb{Z}_p$. This character is an anticyclotomic analogue of χ_{cyc}. Let Λ_a be the regular local ring $\mathcal{O}[[\mathrm{Gal}(K_\infty/K)]]$ endowed with the action of $G_{K,\Sigma}$ coming from the surjection of $G_{K,\Sigma}$ to $\mathrm{Gal}(K_\infty/K)$ and identification of $\mathrm{Gal}(K_\infty/K)$ with $\Lambda_a^\times$. For k an integer and ϵ a finite order character of $\mathrm{Gal}(K_\infty/K)$, an arithmetic point ϕ of weight k and character ϵ of Λ_a is an $\mathcal{O}$-algebra morphism:

$$\begin{aligned}\phi:\ \Lambda_a &\longrightarrow \bar{\mathbb{Q}}_p^\times\\ \gamma &\longmapsto \epsilon(\gamma)\chi_a^k(\gamma)\end{aligned}$$

Let R_{Iw} be the 3-dimensional Gorenstein ring $R[[\mathrm{Gal}(K_\infty/K)]]$. Allowing ourselves a slight abuse of language for which we ask the indulgence of the reader, we define arithmetic points of R_{Iw} to be $\mathcal{O}$-algebra morphisms whose restrictions to R and to Λ_a are arithmetic. Arithmetic primes of R_{Iw} are kernels of arithmetic points. Let $\mathcal{T}_{\mathrm{Iw}}$ be the $G_{K,\Sigma}$-representation $\mathcal{T}\otimes_R R_{\mathrm{Iw}}$ with $G_{K,\Sigma}$ action on both sides of the tensor product.

Specializations of $\mathcal{T}$ Let S be an integral quotient of $R_{\mathrm{Iw}}\otimes_{\mathbb{Z}_p}\overline{\mathbb{Z}}_p$, so in particular S could be R, Λ_a or $\mathcal{O}$. An S-specialization λ of R_{Iw} is an $\mathcal{O}$-algebra map with values in S and an S-regular specialization is an S-specialization whose kernel is generated by a regular sequence. An $S[G_{K,\Sigma}]$-module T_λ is an S-specialization of $\mathcal{T}_{\mathrm{Iw}}$ if T is equal to $\mathcal{T}_{\mathrm{Iw}}\otimes_{R,\lambda} S$ as $S[G_{K,\Sigma}]$-modules. A specialization T_λ is said to be arithmetic if λ is an arithmetic point of R_{Iw}. It is said to contain an arithmetic specialization if T_λ has a quotient which is an arithmetic specialization. In particular, $\mathcal{T}_{\mathrm{Iw}}$, $\mathcal{T}$ and $T\otimes_{\mathcal{O}}\Lambda_a$ all contain arithmetic specializations.

As R is a Gorenstein ring, the same is true for R_{Iw} and thus for S provided there exists an S-regular specialization.

For $v|p$, the local G_{K_v}-representation T_λ is reducible and we define $T^+_{\lambda,v}$ to be $\mathcal{T}^+_{\mathrm{Iw},v}\otimes_{R_{\mathrm{Iw}},\lambda} S$.

§ 2.2. Heegner points and special values of L-function

Rankin-Selberg L-function Let S, λ and V_λ be respectively a discrete valuation ring containing $\mathcal{O}$, an S-valued arithmetic point of R_{Iw} of weight 2 and $T_\lambda\otimes_S \mathrm{Frac}(S)$. Then V_λ is equal to $V(f_\lambda)(1)\otimes\phi^{-1/2}\chi$, where f_λ is the eigenform attached to λ restricted to R, ϕ is the central character of f_λ and χ is a character of $\mathrm{Gal}(K_\infty/K)$. The $G_{K,\Sigma}$-representation V_λ is self-dual and its complex L-function $L(V_\lambda,s)$ coincides with the Rankin-Selberg L-function $L(\pi(f)\times\chi,s+1/2)$. If χ is trivial or has a sufficiently large conductor, our hypotheses on the splitting of primes dividing N implies that $L(V_\lambda,0)$ vanishes at odd, and hence non-trivial, order. As V_λ occurs in the étale cohomology of the curve $X_1(Np^s)$, the Bloch-Kato conjecture then predicts that there are points on $X_1(Np^s)$ rational over a finite sub-extension of K_∞ accounting for this vanishing. This observation remains true if λ restricted to R is fixed, or equivalently if $V(f_\lambda)$ is fixed, and if λ restricted to Λ_a varies. Then, Mazur's conjecture predicts that the order of vanishing of $L(V_\lambda,0)$ is exactly 1 except for a finite number of λ. Symmetrically, we could fix χ and let f_λ vary among arithmetic points of weight 2, in which case R.Greenberg and B.Howard conjecture again that the order of vanishing of $L(V_\lambda,0)$ is exactly 1 except for a finite number of λ. We note that these conjectures are known in most cases thanks to works of V.Vatsal, C.Cornut and B.Howard.

This suggests that there should exist a non-zero p-adic L-function, or rather a leading term of a p-adic L-function, belonging to R_{Iw} and whose image by an arithmetic specialization λ interpolates $L'(V_\lambda,0)/\Omega(\lambda)$ for a suitable choice of period $\Omega(\lambda)$.

Moreover, this p-adic L-function should be linked to the height of families of points rational over sub-extension of K_∞. To the best of knowledge of this author, this p-adic L-function is not known to exist, but he surmises that the p-adic Rankin-Selberg convolution of the p-adic L-function of [1] should satisfy these properties (moreover, the period $\Omega(\lambda)$ would then be an analytic function on λ).

Heegner points Since the seminal work [5], it is known that the special points alluded to in the previous paragraph should be linked with points on $X_1(Np^s)$ parametrizing isogeny between CM elliptic curves, or Heegner points as they have come to be known. In this paragraph, we recall the adelic construction of points in $X_1(Np^s)(K_n)$ verifying distribution relation reminiscing of those we expect of the p-adic L-function. These points were constructed by B.Howard in [8] and he has moreover shown under very mild hypotheses that they are non-torsion if and only if certain L-function does not vanish, as expected.

We consider the tower of Shimura curves $\{X(N,s)\}_{s\geq 1}$ coming from the tower of compact open subgroups $U(N,s)$ of $\mathrm{GL}_2(\mathbb{A}_f)$ defined by:

$$U(N,s)=\left\{g\in\mathrm{GL}_2(\mathbb{A}_f)|g_\ell\equiv\begin{pmatrix}* & *\\ 0 & *\end{pmatrix}\bmod \ell^{\mathrm{ord}_\ell N},\ g_p\equiv\begin{pmatrix}* & *\\ 0 & 1\end{pmatrix}\bmod p^s\right\}$$

The complex points of $X(N,s)$ are given by the double coset:

$$X(N,s)(\mathbb{C})=\mathrm{GL}_2(\mathbb{Q})\backslash(\mathbb{C}-\mathbb{R}\times\mathrm{GL}_2(\mathbb{A}_f)/U(N,s))$$

The embedding of K inside $\mathcal{M}_2(\mathbb{Q})$ defines an action of $K^\times$ on $\mathbb{C}-\mathbb{R}$ which has a unique fixed point Z with positive imaginary part. The set of CM points is the set of complex points $[Z,b]\in X(N,s)(\mathbb{C})$. According to Shimura reciprocity law, CM points are in fact rational over abelian extensions of K. We consider the following family of CM points on the tower $\{X(N,s)\}_{s\geq 1}$:

$$\mathcal{X}(c,s)=\left\{x(c,s)=[Z,b(c,s)]\in X(N,s)|b(c,s)_\ell=\begin{pmatrix}\ell^{\mathrm{ord}_\ell N} & 0\\ 0 & 1\end{pmatrix},\ b(c,s)_p=\begin{pmatrix}p^s & 0\\ 0 & 1\end{pmatrix}\right\}$$

The field of rationality of $x(c,s)$ is denoted by $K(c,s)$, the usual ring-class field of conductor c by $K_0(c)$ and $K_0(c)(\mu_{p^s})$ by $K_0(c,s)$. Let

$$z(c,s)\in H^1(K_0(c),e_{\mathfrak{m}}^{ord}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})(1)\otimes\psi^{-1})$$

be the cohomology class constructed in the following way. First, the Kummer map twisted on the target space and composed with projection on the ordinary part sends $x(c,s)$ to $H^1(K(c,s),e_{\mathfrak{m}}^{ord}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})(1)\otimes\psi^{-1})$. The image $\Phi(x(c,s))$ of

$x(c,s)$ under this map is easily seen to belong to the $\mathrm{Gal}(K_0(c,s)/K_0(cp^s))$-invariants. By purity of $e_{\mathfrak{m}}^{ord}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})$, the Hochschild-Serre spectral sequence for the groups $G_{K_0(c,s)}$ and $G_{K_0(cp^s)}$ induces an isomorphism allowing us to consider $\Phi(x(c,s))$ as a class in $H^1(K_0(cp^s),e_{\mathfrak{m}}^{ord}H^1_{et}(X(N,s)\times_{\mathbb{Q}}\bar{\mathbb{Q}},\mathcal{O})(1)\otimes\chi^{-1})$. Then $z(c,s)$ is equal to $T(p)^{-s}\operatorname{Cor}\Phi(x(c,s))$ where Cor denotes corestriction from $K_0(cp^s)$ to $K_0(c)$. The classes $z(c,s)$ are equivariant under the action of the Hecke algebra and under the action of the Galois groups of ring-class fields in the sense that they satisfy the following Euler system relations.

Proposition 2.2. *[[8] Proposition 2.3.1 and Theorem 3.1.1] Let $\mathscr{L}$ be the set of square-free products of rational primes inert in K with a power of p. Let $c\ell$ be in $\mathscr{L}$, let c^p be the p-power free part of c and let $s'\geq s$ be two integers. Let $\pi_{s'/s}$ be the projection from $X_1(N,s')$ to $X(N,s)$. Then:*

$$\pi_{s'/s}z(c,s')=z(c,s)\tag{2.2}$$

$$\operatorname{Cor}_{K(c\ell)/K(c)}z(cl,s)=T(\ell)z(c,s)\tag{2.3}$$

$$\operatorname{Cor}_{K(c)/K(c^p)}z(c,s)=z(c^p,s)\tag{2.4}$$

In particular, the inverse limit on s and t of $z(cp^t,s)$ composed with corestriction to K defines a class z_∞ in $H^1(K,\mathcal{T}_{\mathrm{Iw}})$. Moreover, the class z_∞ is not R_{Iw}-torsion.

§ 2.3. Selmer complexes and the ETNC

Review of determinants We review briefly the formalism of the determinant functor. Let S be a complete reduced local noetherian ring. A graded invertible S-module is the pair composed of a free S-module of rank 1 and a locally constant function from $\mathrm{Spec}(S)$ to $\mathbb{Z}$. For a finitely generated S-module M free of rank r, the determinant $\det_S M$ of M is the graded invertible S-module $(\bigwedge^r M,r)$. For a bounded complex of free S-modules C, the determinant $\det_S C$ is the graded invertible S-module:

$$\det_S C=\bigotimes_{i\in\mathbb{Z}}\det_S^{(-1)^i}C^i$$

The determinant functor extends to the homotopy category of perfect complexes with morphisms restricted to quasi-isomoprhisms. If the cohomology groups $H^i(C)$ of a bounded complex C regarded as complexes in degree zero are themselves perfect complexes of S-modules, then C is a perfect complex and there is a canonical isomorphism:

$$\det_S C\overset{\sim}{\longrightarrow}\bigotimes_{i\in\mathbb{Z}}\det_S^{(-1)^i}H^i(C)$$

If S is a regular ring and M is a torsion S-module, then M admits finite free resolution by the Auslander-Buchsbaum-Serre theorem so $\det_S M$ is well-defined. As $M\otimes\mathrm{Frac}(S)$

is trivial, $\det_{\mathrm{Frac}(S)} M \otimes \mathrm{Frac}(S)$ is canonically isomorphic to $\mathrm{Frac}(S)$ and the image of $\det_S M$ inside $(\det_S M) \otimes_S \mathrm{Frac}(S)$ can thus be identified with an invertible S-module inside $\mathrm{Frac}(S)$. This module is equal to $(\mathrm{char}_S M)^{-1}$.

Selmer complexes for specializations of $\mathcal{T}_{\mathrm{Iw}}$ Let λ be an S-specialization of $\mathcal{T}_{\mathrm{Iw}}$. We define cohomological complexes which are conjecturally linked to arithmetic properties of T_λ when λ is arithmetic.

Let $C_f(G_{K_v}, T_\lambda)$ be the complex $C^\bullet_{\mathrm{cont}}(\mathrm{Fr}(v), T_\lambda^{I_v})$ for $v \nmid p$ and $C^\bullet_{\mathrm{cont}}(G_{K_v}, T^+_{\lambda,v})$ for $v|p$. There is a morphism i_v from $C_f(G_{K_v}, T_\lambda)$ in $C^\bullet_{\mathrm{cont}}(G_{K_v}, T_\lambda)$ given by inflation if $v \nmid p$ and by the inclusion of $T^+_{\lambda,v}$ inside T_λ if $v|p$.

Definition 2.3. Let $\mathrm{R}\Gamma_f(G_{K,\Sigma}, T_\lambda)$ be the object in the derived category corresponding to

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(G_{K,\Sigma}, T_\lambda) \oplus \bigoplus_{v\in\Sigma} C_f(G_{K_v}, T_\lambda) \xrightarrow{\mathrm{res}_v - i_v} \bigoplus_{v\in\Sigma} C^\bullet_{\mathrm{cont}}(G_{K_v}, T_\lambda)\right)[-1]$$

and let $\tilde{H}^i_f(G_{K,\Sigma}, T_\lambda)$ be its i-th cohomology group.

The S-module T_λ is free and $G_{K,\Sigma}$ and the G_{K_v} have finite cohomological p-dimension uniformly bounded so the complex $\mathrm{R}\Gamma_f(G_{K,\Sigma}, T_\lambda)$ is a perfect complex of S-modules acyclic outside $[0, 3]$. The S-module $\tilde{H}^0_f(G_{K,\Sigma}, T_\lambda)$ injects in $H^0(G_{K,\Sigma}, T_\lambda)$ so is trivial by absolute irreducibility of $\bar{\rho}$. Hence, the same is true of $\tilde{H}^3_f(G_{K,\Sigma}, T_\lambda)$ by self-duality of T_λ and [11] statement (8.9.10).

When $H^0(G_{K_v}, T^-_{\lambda,v})$ is zero for all $v|p$, the long exact sequence in cohomology induced by definition 2.3 shows that $\tilde{H}^1_f(G_{K,\Sigma}, T_\lambda)$ is equal to the usual compact Greenberg Selmer group:

$$H^1_{\mathrm{Gr}}(G_{K,\Sigma}, T_\lambda) = \ker\left(H^1(G_{K,\Sigma}, T_\lambda) \longrightarrow \bigoplus_{v\in\Sigma} H^1(G_{K,\Sigma}, T_\lambda)/H^1_{\mathrm{Gr}}(G_{K_v}, T_\lambda)\right)$$

Here $H^1_{\mathrm{Gr}}(K_v, T_\lambda)$ is equal to $H^1(K_v^{ur}, T_\lambda)$ if $v \nmid p$ and to $H^1(K_v, T^+_{\lambda,v})$ if $v|p$. We note that $H^0(G_{K_v}, T^-_{\lambda,v})$ is zero in particular if λ is arithmetic and if T_λ is a potentially crystalline G_{K_v}-representation, or if it is of weight different from 2. Hence, $H^0(G_{K_v}, \mathcal{T}^-)$ and $H^0(G_{K_v}, \mathcal{T}^-_{\mathrm{Iw}})$ are trivial. A specialization λ is said to be exceptional if there exists a finite extension L_w of K_v such that $H^0(G_{L_w}, T^-_\lambda)$ is not trivial.

When S is a discrete valuation ring, let $\mathrm{R}\Gamma_c(\mathrm{Spec}\,\mathcal{O}_K[1/\Sigma], T_\lambda)$ be the object in the derived category corresponding to

$$\mathrm{Cone}\left(C^\bullet_{\mathrm{cont}}(\mathrm{Spec}\,\mathcal{O}_K[1/\Sigma], T_\lambda) \xrightarrow{\mathrm{res}_v - i_v} \bigoplus_{v\in\Sigma} C^\bullet_{\mathrm{cont}}(G_{K_v}, T_\lambda)\right)[-1]$$

Special classes in $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$ According to proposition 2.2, the class z_∞ belongs to $H^1(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$.

Proposition 2.4 ([8] Proposition 2.4.5)**.** *For all $c\in\mathscr{L}$, the class $z_\infty(c)$ belongs to $\tilde{H}^1_f(G_{K(c),\Sigma},\mathcal{T}_{\mathrm{Iw}})$. The class z_∞ belongs to $\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$.*

If λ is an S-specialization which is not exceptional, the natural map from $H^1(K,\mathcal{T}_{\mathrm{Iw}})$ to $H^1(K,T_\lambda)$ induced by λ defines a class z_λ which belongs to $H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)$, and hence to $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$. Even if λ is exceptional, the class z_λ belongs to $H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)$. The short exact sequence

$$0\longrightarrow\bigoplus_{v|p}H^0(G_{K_v},T^-_{\lambda,v})\longrightarrow\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)\longrightarrow H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)\longrightarrow 0$$

and the fact that the inverse limit on n of $H^0(G_{K(n)_v},T^-_{\lambda,v})$ vanishes establish an isomorphism between $\tilde{H}^1_f(G_{K_\infty,\Sigma},T_\lambda)$ and $H^1_{\mathrm{Gr}}(G_{K_\infty,\Sigma},T_\lambda)$. As z_λ belongs to the image of this latter group inside $H^1_{\mathrm{Gr}}(G_{K,\Sigma},T_\lambda)$, it admits a canonical lift to $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$.

Thanks to the following results of B.Howard, we very often know that z_λ is not S-torsion.

Proposition 2.5. *[[8] theorem 3.1.1 and [9] corollary 5] Let λ be an arithmetic point of R. Let T be $\mathcal{T}_{\mathrm{Iw}}$ or $T_\lambda\otimes_S\Lambda_a$. Then the image of z_∞ in $\tilde{H}^1_f(G_{K,\Sigma},T)$ is not torsion. Assume that there exists an arithmetic point μ of R of weight 2 such that $L'(V_\mu,0)$ does not vanish. Then the image of z_∞ in $\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T})$ is not torsion and z_λ is not torsion for almost all arithmetic points λ of R. The set of specialization of R_{Iw} such that z_λ is torsion is of codimension at least 1.*

An Equivariant Tamagawa Number Conjecture When S is a discrete valuation ring and λ is arithmetic, the Tamagawa Number Conjecture predicts that there exists an isomorphism of free S-modules

$$\zeta_{\lambda,c}:\ S\longrightarrow\det{}^{-1}_S\mathrm{R}\Gamma_c(\operatorname{Spec}\mathcal{O}_K[1/\Sigma],T_\lambda)$$

such that $\zeta_{\lambda,c}\otimes_S\mathrm{Frac}(S)$ expresses the p-part of the algebraic part of the leading term of the complex L-function of V_λ. If this isomorphism exists, then it can be modified to give an isomorphism of free S-modules:

$$\zeta_{\lambda,f}:\ S\longrightarrow\det{}^{-1}_S\mathrm{R}\Gamma_f(G_{K,\Sigma},T_\lambda)$$

The point of this modification is to link $\zeta_{f,\lambda}$ with the value of the p-adic L-function rather than with the complex L-function. The philosophy of the ETNC for the family of motives $\{h^1(X_1(Np^s))\otimes h^0(\mathrm{Spec}(K_n))\}_{s,n}$ predicts that there exists an isomorphism of free R_{Iw}-modules

$$\zeta_f:\ R_{\mathrm{Iw}}\longrightarrow \det{}^{-1}_{R_{\mathrm{Iw}}}\mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$$

verifying the change of ring property that $\zeta_f\otimes_{R_{\mathrm{Iw}},\lambda}S$ is equal to $\zeta_{\lambda,f}$ and such that $\zeta_f\otimes\mathrm{Frac}(R_{\mathrm{Iw}})$ is linked with the p-part of the algebraic part of the leading term of the conjectural p-adic L-function L_p.

Without assuming any conjecture, we know there is a non-torsion element z_∞ inside $\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$. The isomorphism

$$\tilde{H}^1_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})\overset{\sim}{\longrightarrow}\mathrm{Hom}_{R_{\mathrm{Iw}}}(\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})$$

defines a non-zero element z^*_∞ inside $\mathrm{Hom}_{R_{\mathrm{Iw}}}(\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})\otimes\mathrm{Frac}(R_{\mathrm{Iw}})$. Let $d\in\mathrm{Frac}(R_{\mathrm{Iw}})$ and $z'_\infty\in\mathrm{Hom}_{R_{\mathrm{Iw}}}(\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})$ be such that $z^*_\infty=d^{-1}z'_\infty$ in $\mathrm{Hom}_{R_{\mathrm{Iw}}}(\tilde{H}^2_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})$. Multiplications by z_∞, z'_∞ and d define morphisms of complexes:

$$R_{\mathrm{Iw}}[-1]\xrightarrow{z_\infty}\mathrm{R}\,\Gamma_f(G_{K,\Sigma},T_\lambda)$$

$$R_{\mathrm{Iw}}[-2]\xrightarrow{z'_\infty}\mathrm{R}\,\mathrm{Hom}(\mathrm{R}\,\Gamma_f(G_{K,\Sigma},T_\lambda),R_{\mathrm{Iw}})$$

$$R_{\mathrm{Iw}}\xrightarrow{d}R_{\mathrm{Iw}}$$

Let $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ be the product of the determinants of the cones of these morphisms of complexes. Then $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ does not depend on the choice of d and z'_∞. The isomorphism

$$\mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})\overset{\sim}{\longrightarrow}\mathrm{R}\,\mathrm{Hom}_{R_{\mathrm{Iw}}}(\mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}}),R_{\mathrm{Iw}})\otimes\mathrm{Frac}(R_{\mathrm{Iw}})[-3]$$

of [11] theorem 8.9.11 induces an isomorphism from $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}}}\mathrm{Frac}(R_{\mathrm{Iw}})$ to $\mathrm{Frac}(R_{\mathrm{Iw}})$, which we take to be an indentification in all that follows. Let $\mathscr{I}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ be the image of $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ in $\mathrm{Frac}(R_{\mathrm{Iw}})$ induced by this identification.

More generally, we define in the same way $\mathscr{X}_f(T_\lambda,S)$ and $\mathscr{I}_f(T_\lambda,S)$ for all S-valued λ with S a Gorenstein domain and z_λ not S-torsion. When in addition S is a discrete valuation ring and λ is a non-exceptional S-specialization, the method of Euler systems shows that $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$ is free of rank 1 and hence that $\tilde{H}^2_f(G_{K,\Sigma},T_\lambda)$ is of rank 1. The isomorphism between $\mathscr{X}_f(T_\lambda,S)\otimes\mathrm{Frac}(S)$ and $\mathrm{Frac}(S)$ then identifies $\mathscr{I}_f(T_\lambda,S)$ with:

$$\frac{|\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)/z_\lambda|^2}{|\tilde{H}^2_f(G_{K,\Sigma})_{\mathrm{tors}}|}S$$

The TNC suggests that this module is included in S, and is equal to S if z_λ is sufficiently well optimized. Generalizing the case of discrete valuation ring, we are led to the following two conjectures for a specialization λ containing an arithmetic point with values in a Gorenstein domain S and such that z_λ is not S-torsion.

Definition 2.6. Let $\mathbf{Conj}(\lambda, S)$ be the statement that the image of $\mathscr{I}_f(T_\lambda, S)$ is included in S.

Definition 2.7. Let $\mathbf{StrongConj}(\lambda, S)$ be the statement that $\mathscr{I}_f(T_\lambda, S)$ is equal to S.

We draw the attention of the reader to the fact that if T_λ is an arithmetic specialization, one can consider both $\mathbf{Conj}(T_\lambda, S)$ with S equal to a discrete valuation ring or to $\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}(N, s)$. The former case is algebraically simpler, but in the latter λ is a regular specialization. When R itself is a regular ring, then $\mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}(N, s)$ is also regular by proposition 2.1 and this distinction disappears. We consider especially the following special cases of conjectures 2.6 and 2.7, in which we implicitly conjecture that z_λ is not torsion: $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$, $\mathbf{Conj}(\mathcal{T}, R)$, $\mathbf{Conj}(T \otimes \Lambda_a, \mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}(N, s) \otimes \Lambda_a)$, $\mathbf{Conj}(T \otimes \Lambda_a, S \otimes \Lambda_a)$, $\mathbf{Conj}(T, \mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}(N, s))$ and $\mathbf{Conj}(T, S)$. When λ is arithmetic of weight 2, $\mathbf{StrongConj}(T_\lambda, \Lambda_a)$ is a conjecture of B.Perrin-Riou. When R is regular, $\mathbf{StrongConj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ is a conjecture of B.Howard.

The author confesses that he is inclined to a certain dose of scepticism towards $\mathbf{StrongConj}(\lambda, S)$ as stated, if only because he feels the question of whether Heegner points correspond to improved or to usual p-adic L-function has not been sufficiently explored in what precedes.

§ 2.4. Main theorem

We now state the two main theorems of this article.

Theorem 2.8. *If $\boldsymbol{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ is true, then $\boldsymbol{Conj}(\lambda, S)$ is true for all regular specializations λ containing an arithmetic specialization and such that z_λ is not torsion, and in particular for $\boldsymbol{Conj}(\mathcal{T}, R)$, $\boldsymbol{Conj}(T \otimes \Lambda_a, \Lambda_a)$ and $\boldsymbol{Conj}(T, \mathbf{T}^{\mathrm{ord}}_{\mathfrak{m}}(N, s))$ provided z is not torsion for these specializations. Under the same hypothesis, if there exists a regular specialization λ containing an arithmetic specialization such that $\boldsymbol{StrongConj}(\lambda, S)$ is true, then $\boldsymbol{StrongConj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ and $\boldsymbol{StrongConj}(T_\lambda, S)$ for all regular λ containing an arihtmetic specialization and such that $z_\lambda$is not torsion are also true.*

Theorem 2.9. *Let T_λ be an arithmetic specialization of $\mathcal{T}$ with coefficients in the discrete valuation ring S. Then $\boldsymbol{Conj}(T_\lambda \otimes_S \Lambda_a, \Lambda_a)$ is true and $\boldsymbol{Conj}(T_\lambda, S)$ is true provided z_λ is not torsion. Assume that R is a regular ring. Then $\boldsymbol{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$ is true and $\boldsymbol{Conj}(\mathcal{T}, R)$ is true provided z_λ is not torsion.*

Theorem 2.8 roughly states that our conjectures satisfy the change of rings property, while theorem 2.9 corresponds in usual cases to a divisibility of characteristic ideals in the Iwasawa Main Conjecture. Because of their similar statement apart from increasing generality, the reader might believe that $\mathbf{Conj}(T,S)$, $\mathbf{Conj}(T\otimes\Lambda_a,\Lambda_a)$, $\mathbf{Conj}(\mathcal{T},R)$ and $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ are proved in that order and by roughly the same method. In fact, this is far from true: we first establish a weaker version of $\mathbf{Conj}(T,S)$ for many non-necessarily arithmetic specializations T, then $\mathbf{Conj}(T\otimes\Lambda_a,\Lambda_a)$ for the same set of T, then $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ and only at that point $\mathbf{Conj}(T,S)$ and $\mathbf{Conj}(\mathcal{T},R)$ using theorem 2.8.

§ 3. Proofs of theorems 2.8 and 2.9

§ 3.1. Control theorem for Selmer complexes

Proposition 3.1. *If λ is an S-specialization of R_{Iw} containing an arithmetic point, then:*

$$\mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda}S\overset{\sim}{\longrightarrow}\mathrm{R}\,\Gamma_f(G_{K,\Sigma},T_\lambda)$$

Proof. It is enough to prove a comparable base-change statement for the complexes involved in the definition of $\mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})$. As λ is a regular specialization by proposition 2.1 and the fact that Λ_a is a regular ring, the following base-change results follow from the fact that taking continuous cochains is an exact functor:

$$C^\bullet_{\mathrm{cont}}(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda}S\overset{\sim}{\longrightarrow}C^\bullet_{\mathrm{cont}}(G_{K,\Sigma},T_\lambda)$$
$$C^\bullet_{\mathrm{cont}}(G_{K_v},\mathcal{T}_{\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda}S\overset{\sim}{\longrightarrow}C^\bullet_{\mathrm{cont}}(G_{K_v},T_\lambda)$$
$$C^\bullet_{\mathrm{cont}}(G_{K_v},\mathcal{T}^+_{\mathrm{Iw},v})\overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda}S\overset{\sim}{\longrightarrow}C^\bullet_{\mathrm{cont}}(G_{K_v},T^+_v)$$

To conclude, it is thus enough to prove that $C_f(G_{K_v},\mathcal{T}_{\mathrm{Iw}})$ satisfies the same base-change property for $v|N$ and this in turn would follow provided that:

$$\mathcal{T}_{\mathrm{Iw}}^{I_v}\otimes_{R_{\mathrm{Iw}},\lambda}S\overset{\sim}{\longrightarrow}T_\lambda^{I_v}$$

This comes from the fact that the action of G_{K_v} on Λ_a is unramified and that the automorphic type of all arithmetic points of R are the same at v. See [4]. □

Let λ be a regular S-specialization of R_{Iw} such that z_λ is not torsion. According to the preceding proposition, the complex $\mathrm{R}\,\Gamma_f(G_{K,\Sigma},\mathcal{T}_{\mathrm{Iw}})\overset{\mathrm{L}}{\otimes}_{R_{\mathrm{Iw}},\lambda}S$ is isomorphic to $\mathrm{R}\,\Gamma_f(G_{K,\Sigma},T_\lambda)$. By construction $\mathscr{X}_f(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})\otimes_{R_{\mathrm{Iw}},\lambda}S$ is thus equal to $\mathscr{X}_f(T_\lambda,S)$. As an element of R_{Iw} specializes to an element of S and to an element of $S^\times$ if and only if it is in $R_{\mathrm{Iw}}^\times$, theorem 2.8 follows.

§ 3.2. The method of Euler systems

Assume in this sub-section that S is a discrete valuation ring and that λ is an S-valued non exceptional S-specialization of R_{Iw} such that z_λ is not S-torsion. Note that we do not assume λ to be arithmetic.

Proposition 3.2. *The S-module $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)$ is free of rank 1 and $\tilde{H}^2_f(G_{K,\Sigma})$ is of rank 1. Let $C(\lambda)$ be:*

$$C(\lambda)=\left(\prod_{v|N}|H^1(G_{K_v},T_\lambda)/H^1(\mathrm{Fr}(v),T_\lambda^{I_v})|\right)^2\left(\prod_{v|p}|H^1(G_{K_v},T_\lambda^-)_{\mathrm{tors}}|\right)^2$$

Then:

$$|\tilde{H}^2_f(G_{K,\Sigma},T_\lambda)|\leq C(\lambda)|\tilde{H}^1_f(G_{K,\Sigma},T_\lambda)/z|^2 \tag{3.1}$$

Equivalently, $\mathscr{I}_f(T_\lambda,S)$ is included in $C(\lambda)^{-1}S$.

Proof. This is a slightly non-standard presentation of the standard results given by the method of Euler systems on bound of Selmer groups, which we obtain by the method of [10, 6]. In order to apply the method of Euler system in the setting of CM points, it is often assumed that the image of ρ_λ is large enough. We observe that in the axiomatic of [6], this is used only in lemma 1.6.2, and hence only via $\bar{\rho}_\lambda$, which does not depend on λ. As we have assumed that $\bar{\rho}_f$ is absolutely irreducible after restriction to G_K, the proof of lemma 1.6.2 carries without modification. □

Let λ be a specialization of $T\otimes_{\mathcal{O}}\Lambda_a$ with value in a discrete valuation ring S. Then $T_\lambda^{I_v}$ is equal to $(T^{I_v}\otimes\Lambda)\otimes_{\mathcal{O}}S$ for $v\nmid p$ and $|H^1(G_{K_v},T_\lambda^-)_{\mathrm{tors}}|$ is bounded by $|H^0(G_{K_{\infty,v}},T^-\otimes\mathrm{Frac}(S)/S)|$. As long as S is fixed, $C(\lambda)$ is thus bounded independently of λ. We will use proposition 3.2 to show that $\mathbf{Conj}(T_\lambda,S)$ is uniformly not too false when λ varies in the specializations of $T\otimes_{\mathcal{O}}\Lambda_a$.

§ 3.3. Descent for Selmer complexes

We assume henceforth R_{Iw} to be regular. Let S be a regular specialization of R_{Iw} of Krull dimension at least 2, T an S-specialization of $\mathcal{T}_{\mathrm{Iw}}$ and x an S-regular element. Then, the counterpart for x of proposition 3.1 is not true because T^{I_v}/xT^{I_v} is not in general isomorphic to $H^0(I_v,T/xT)$ for $v|N$. However, as the S-modules $H^1(G_{K_v},T)_{\mathrm{tors}}$ are of finite type, there exists a power p^{n_0} such that for all $n\geq n_0$ and for all $v|N$, $H^1(G_{K_v},T)[x+p^n]$ is p-torsion. Hence, there exists an integer M such that for all $n\geq n_0$ and for all $v|N$, the image of

$$\prod_{v|N}\det{}_{S/x}H^1(G_{K_v},T)[x+p^n]$$

inside $\operatorname{Frac}(S)$ is included in $p^M S$. Consequently, we have the following proposition, which is very useful to relate $\mathbf{Conj}(T,S)$ to $\mathbf{Conj}(T/x,S/x)$.

Proposition 3.3. *Let X be a non-zero element of the maximal ideal $\mathfrak{m}_S$ of S. Assume that the image of $\mathscr{I}_f(T,S)$ inside* $\operatorname{Frac}(S)$ *is not contained in $S[1/X]$. Then for all $M\in\mathbb{N}$ there exists a p-adically dense set of regular elements Reg such that if x belongs to Reg, then $x\notin\mathfrak{m}_S^2$ and $\mathscr{I}_f(T/x,S/x)$ is not included in S. If S is of dimension 2, then in addition X is not zero in S/x and $\mathscr{I}_f(T/x,S/x)$ is not in $X^{-M}S/x$.*

Proof. After a finite flat extension if necessary, we consider S as a power-series ring. Then $\mathscr{I}_f(T,S)$ is equal to a fractional ideal (P/Q) and Q is not a unit by assumption.

Assume first that S is of dimension 3. Using Weierstrass preparation theorem, we easily see that there exists $x\in\mathfrak{m}_S\backslash\mathfrak{m}_S^2$ such that the image of $\mathscr{I}_f(T,S)\otimes_S S/x$ inside $\operatorname{Frac}(S/x)$ is not in $(S/x)[1/X]$.

Now S is of dimension 2. If Q is not a unit in $S\otimes\bar{\mathbb{Q}}_p$, then specializing close enough to a zero of Q shows that for all M, there exists x satisfying the requirement of the proposition except the last and such that $\mathscr{I}_f(T,S)$ is not in $p^{-M}S/x$. But according to the discussion preceding the proposition and changing slightly x if necessary, there exists a fixed M_0 such that $\mathscr{I}_f(T/x,S/x)$ is in $p^{M_0}\mathscr{I}_f(,S)\otimes S/x$. Hence there exists an x satisfying the requirements of the proposition. If now Q is a unit in $\bar{\mathbb{Q}}_p$, the same proof interchanging the role of p and X establishes the proposition. □

We now finish the proof of theorem 2.9. Let T_λ be a non-exceptional S-valued specialization of $\mathcal{T}_{\mathrm{Iw}}$ with S a discrete valuation ring such that z is not torsion as an element of $\tilde{H}^1_f(G_{K,\Sigma},T_\lambda\otimes\Lambda_a)$. According to proposition 2.5, this is true for a p-adically dense subset of λ with values in S and for all such arithmetic λ. If $T_\lambda\otimes_S\Lambda_a$ violates $\mathbf{Conj}(T_\lambda\otimes_S\Lambda_a,\Lambda_a)$, then by proposition 3.3 there exists a specialization μ such that z_μ is not torsion and such that (3.1) is false. This is a contradiction with proposition 3.2 so $\mathbf{Conj}(T_\lambda\otimes_S\Lambda_a,\Lambda_a)$ is true, and $\mathbf{Conj}(T_\lambda,S)$ follows by theorem 2.8. If $\mathcal{T}_{\mathrm{Iw}}$ violates $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$, then by proposition 3.3, there exists an infinite number of λ with values in S such that $\mathbf{Conj}(T_\lambda\otimes_S\Lambda_a,\Lambda_a)$ is false, and this contradicts our previous result. So $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}},R_{\mathrm{Iw}})$ is true, and thus $\mathbf{Conj}(\mathcal{T},R)$ is also true by theorem 2.8 provided z is not R-torsion.

§ 4. Perspectives

§ 4.1. Leading term of p-adic L-function and p-adic height pairing

We observe that the knowledge of the non-torsion class z_λ induces a trivialization of $\det_S \mathrm{R}\Gamma_f(G_{K,\Sigma},T_\lambda)$, and hence allows a formulation of a variant of the ETNC, even though the p-adic L-function vanishes and even though we do not assume that the p-adic height pairing is not degenerate.

When the p-adic height pairing is known to be non-degenerate and z_λ is not torsion, then the convoluted construction of $\mathscr{X}_f(T_\lambda, S)$ can be replaced by the product of the determinants of the following complexes:

$$\mathrm{Cone}(S[-1] \xrightarrow{z_\lambda} \mathrm{R}\,\Gamma_f(G_{K,\Sigma}, T_\lambda))$$
$$\mathrm{Cone}(S[-1] \xrightarrow{h(z_\lambda,\cdot)} \mathrm{R}\,\mathrm{Hom}_S(\mathrm{R}\,\Gamma_f(G_{K,\Sigma}, T_\lambda), S))$$

Interpreting $\mathscr{X}_f(T_\lambda, S)$ as the determinant of the cone of the morphism

$$S[-1] \xrightarrow{h(z,z)} S[-1]$$

associates $h(z,z)$ with an element of S which we naturally conjecture to be the algebraic part of $\lambda(L_p(\mathcal{T}_{\mathrm{Iw}}))$. Here again, the author would like to express his scepticism about the literal truth of the previous statement as he feels the issue of whether it is the improved or standard p-adic L-function which appears has not been enough explored in what precedes.

§ 4.2. Totally real field F and quaternion algebras

Theorems 2.8 and 2.9 can be generalized fairly well to nearly ordinary automorphic representations of the multiplicative group of a quaternion algebra over a totally real field such that at most one infinite place does not ramify, though several hurdles appear in this more general setting.

In the indefinite case, we refer the reader to [3] and explain here the most serious difficulties. First, one replaces the tower of modular curves $X_1(Np^s)$ by a tower of compact Shimura curves $X(s)$. However, as the q-expansion principle is then lacking, the freeness of $e_{\mathfrak{m}}^{ord} H^1_{et}(X(1) \times_F \bar{F}, \mathcal{O})$ over the Hecke algebra is in general not known (it is typically known under some conditions on $\bar{\rho}_f$, in which case it follows from the Taylor-Wiles machinery). Considering general quaternion algebra allows for more supple choices of N and K, as the construction of CM points does not require that all primes dividing N split in K, as in [7, 2]. But under these more general conditions, the class z_λ is not known to belong to $H^1_f(G_{K_v}, T_\lambda)$ at places v dividing N and inert in K. Consequently, we are only able to prove that $\mathscr{I}_f(T_\lambda \otimes_S \Lambda_a, T_\lambda \otimes_S \Lambda_a)$ is in $\Lambda_a[1/p]$ but from this, we are unfortunately not able to prove $\mathbf{Conj}(\mathcal{T}_{\mathrm{Iw}}, R_{\mathrm{Iw}})$.

Recent and forthcoming works of M.Longo and S.Vigni treat simultaneously the definite and indefinite case over $\mathbb{Q}$.

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