

# Control theorems for Selmer groups of nearly ordinary deformations

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## 1 Introduction

Let  $(\mathcal{T}, \rho, R)$  (see section 1.1 below for that notation) be a  $p$ -adic family of rank 2 Galois representations, this is to say a free module of rank 2 over a complete local noetherian ring  $R$  of residual characteristic  $p$  with a continuous action of the absolute Galois group  $G_F$  of a totally real number field  $F$ . We assume that  $(\mathcal{T}, \rho, R)$  satisfies the following conditions:

1. The action of  $G_F$  on  $\mathcal{T}$  is unramified outside a finite set of places  $\Sigma$ .
2. For all  $v|p$ , the restriction of  $\rho$  to  $G_{F_v}$  is reducible and .
3. There exists a Zariski-dense subset  $S$ , called the set of arithmetic points, of  $\text{Hom}(R, \bar{\mathbb{Q}}_p)$  such that  $V_\kappa = (\mathcal{T} \otimes_R \kappa(R)) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$  is the  $p$ -adic étale realization of a pure critical motive  $M_\kappa$  for all  $\kappa \in S$ .
4. There exists  $\kappa \in S$  such that the  $L$ -function  $L(M_\kappa, s)$  of  $M_\kappa$  does not vanish at  $s = 0$

Under these hypotheses, it was conjectured in [Gre91, Gre94] (see also [Och06]) that there exists a non zero-divisor  $p$ -adic  $L$ -function  $L_p(\mathcal{T}) \in R$  interpolating the critical values of the  $L$ -function  $L(M_\kappa, s)$  in the sense that  $\kappa(L_p(\mathcal{T}))$  is equal to  $L(M_\kappa, 0)$  divided by a suitable period and multiplied by Euler factors for all  $v|p$ . On the other hand, properties 1 and 2 and Galois cohomology make possible the construction of a finite type  $R$ -module  $\text{Sel}(F, \mathcal{T})$  and, for all  $\kappa \in S$  and all  $\kappa(R)$ -lattice  $T_\kappa$  inside  $V_\kappa$ , of finite type  $\kappa(R)$ -modules  $\text{Sel}(F, T_\kappa)$ . These modules are conjectured to be linked respectively with  $L_p(\mathcal{T})$  and  $\kappa(L_p(\mathcal{T}))$ . In particular, if  $L(M_\kappa, 0)$  is non-zero, then it is conjectured in [BK90] that  $\text{Sel}(F, V_\kappa)$  is trivial, or equivalently that  $\text{Sel}(F, T_\kappa)$  is torsion. In analogy with the case of  $R = \mathbb{Z}_p$  and taking into account property 4 above, the  $R$ -module  $\text{Sel}(F, \mathcal{T})$  is conjectured to be torsion. More precisely, the Tamagawa Number Conjecture states that when  $\kappa(L_p(\mathcal{T}))$  is non-zero, its  $p$ -part should be related to the length of  $\text{Sel}(F, T_\kappa)$ . When  $R$  is a normal ring, the Iwasawa Main Conjecture states that the characteristic ideal  $\text{char}_R \text{Sel}(F, \mathcal{T})$  should be equal to the ideal generated by  $L_p(\mathcal{T})$ .

Combining these two conjectures with the interpolation property defining  $L_p(\mathcal{T})$  suggests that  $\text{Sel}(F, T_\kappa)$  and  $\text{Sel}(F, \mathcal{T}) \otimes_R \kappa(R)$  should both be linked with  $\kappa(L_p(\mathcal{T}))$  and thus that the natural map

$$\text{Sel}(F, \mathcal{T}) \otimes_R \kappa(R) \longrightarrow \text{Sel}(F, T_\kappa)$$

should be very close to an isomorphism. The fact that Selmer groups indeed satisfy this property is called a control theorem.

Historically, the first setting in which such a theorem was proved was in [Maz72] (in fact even before  $p$ -adic  $L$ -function of abelian varieties were investigated but with the ideas of Iwasawa in mind)

for the family of  $G_F$ -representations  $T_p A \otimes_{\mathbb{Z}_p} \Lambda$ , where  $T_p A$  is the Tate module of a good ordinary abelian variety defined over  $F$  and  $\Lambda = \mathbb{Z}_p[[\text{Gal}(F_\infty/F)]]$  is the complete group algebra of the Galois group of the cyclotomic  $\mathbb{Z}_p$ -extension of  $F$  with its natural  $\text{Gal}(F_\infty/F)$ -action. Subsequently, it was generalized to abelian varieties and more general  $p$ -adic Lie extensions (see among others [Gre03]), to  $\mathbb{Z}_p'$ -extensions for some abstract  $R[G_F]$ -modules in [Nek06, (8.10)] and to families of  $G_{\mathbb{Q}}$ -representations arising from nearly ordinary Hida families of modular forms in [Och01, Och06].

From now on, we assume that  $\mathcal{T}$  arises from Hida theory of nearly ordinary automorphic representations of  $\text{GL}_2(\mathbb{A}_F)$  in the following sense. In [Hid88], nearly ordinary Hecke algebras  $\mathbf{T}^{\text{ord}}$  for inner forms of  $\text{GL}_2$  were studied and were shown to be noetherian rings of dimension at most  $2 + [F : \mathbb{Q}] + \delta_{F,p}$  where  $\delta_{F,p}$  is the defect of Leopoldt's conjecture at  $p$  and finite torsion-free modules over a regular ring  $\Lambda$ . Let  $R = \mathbf{T}_m^{\text{ord}}/\mathfrak{a}$  be the quotient of a local factor of  $\mathbf{T}^{\text{ord}}$  by a minimal prime  $\mathfrak{a}$ . It is a local domain with residual field  $\mathbb{F}$ . To  $R$  is attached a residual  $G_F$ -representation  $(\bar{T}, \bar{\rho}_m, \mathbb{F})$  with coefficients in  $\mathbb{F}$ . For simplicity of exposition, we assume in this introduction that  $\bar{\rho}_m$  is irreducible and that  $R$  is a regular ring (much of our results in fact require need neither assumptions, see section 2.1.3 and propositions 3.1 and 3.2). In [Wil88, Hid89], it is explained how to construct a  $G_F$ -representations  $(T, \rho, R)$  satisfying the properties 1-3 listed above. In particular, to an arithmetic point  $\kappa$  of  $\mathcal{T}$  is attached a newform  $f$  which is an eigenform under the action of  $R$  and  $\kappa$  is the map which sends a Hecke operator to the corresponding eigenvalue of  $f$ . We remark here that families of  $G_F$ -representation satisfying the properties 1-3 should in fact all arise in this way according to [FM95].

In this article, we consider the Pontryagin dual  $\text{Sel}(F, \mathcal{T})$  of the Selmer group of [Gre91] as well as the second cohomology group  $\tilde{H}_f^2(F, \mathcal{T})$  of the Selmer complex  $\text{R}\Gamma_f(F, \mathcal{T})$ , as defined and studied in [Nek06], and prove control theorems for them. We remark that both objects and thus both style of control theorems have their use: Selmer groups are easier to link to special values of  $L$ -function whereas Selmer complexes satisfy convenient base change properties. The following theorem summarizes our results but we refer to section 2.2 for definitions and to propositions 3.1 and 3.2 for precise statements (the reader is especially advised that the hypothesis that  $R$  is a regular ring and that  $\bar{\rho}_m$  is irreducible is not used in proposition 3.1).

**Theorem 1.** *Let  $R$  be an irreducible component of the nearly ordinary Hecke algebra localized at a non-Eisenstein maximal ideal and let  $(T, \rho, R)$  be the  $G_F$ -representation with coefficients in  $R$  constructed by Hida theory. We assume that  $R$  is a regular ring. Let  $\kappa$  be an arithmetic point of  $R$  with values in a discrete valuation ring  $\mathcal{O}$ . Let  $\pi(f_\kappa)$  be the automorphic representation attached to  $\kappa$  and  $T(f_\kappa)$  be the  $G_F$ -representation with coefficients in  $\mathcal{O}$  attached to  $f_\kappa$ . The second cohomology group  $\tilde{H}_f^2(F, \mathcal{T})$  of  $\text{R}\Gamma_f(F, \mathcal{T})$  satisfies:*

$$\tilde{H}_f^2(F, \mathcal{T}) \otimes_{R, \kappa} \mathcal{O} \xrightarrow{\sim} \tilde{H}_f^2(F, T(f_\kappa)) \quad (1.0.1)$$

Assume that  $\pi(f_\kappa)_v$  is not a Steinberg representation at  $v \nmid p$ . Then:

$$\text{Sel}_{\text{str}}(F, \mathcal{T}) \otimes_{R, \kappa} \mathcal{O} \xrightarrow{\sim} \text{Sel}_{\text{str}}(F, T(f_\kappa))$$

Moreover, the surjection

$$\text{Sel}(F, \mathcal{T}) \otimes_{R, \kappa} \mathcal{O} \twoheadrightarrow \text{Sel}(F, T(f_\kappa))$$

has only finite kernel.

The classical lower bound for the domain of convergence of  $L(\pi(f_\kappa), s)$  due to Hecke shows that there exists arithmetic specializations  $V_\kappa$  such that  $L(V_\kappa, 0)$  is non-zero, and in fact that they form a Zariski-dense subset. Consequently, the Selmer groups we consider are expected to be torsion, and this is indeed known when  $F = \mathbb{Q}$  by [Kat04, Theorem 14.2] and [Och01]. When  $F \neq \mathbb{Q}$ , however, and perhaps to the exception of families with complex multiplication, even examples of

this were yet unknown to the best of knowledge of the authors. Our control theorem shows that if there exists an arithmetic  $\kappa$  such that  $\text{Sel}(F, V_\kappa)$  vanishes, then  $\text{Sel}(F, \mathcal{T})$  is torsion. In proposition 4.1, we give sufficient conditions for this to happen. These conditions are verified for roughly half of the nearly ordinary families coming from Hida theory so that though our work is far from settling this question, it provides plenty of examples. Finally, we show that our control theorems imply that the Iwasawa Main Conjecture for  $\mathcal{T}$  implies the Tamagawa Number Conjecture for an infinite number of arithmetic specializations of  $\mathcal{T}$ .

This article is organized as follows. First, we recall the construction and properties of  $G_F$ -representations with coefficients in Hecke algebra coming from Hida theory and introduce Selmer groups and Selmer complexes attached to them. The control theorems proved in the second part follows from three main ideas: the local regularity of Hecke rings in the neighborhood of arithmetic points, the base-change properties of complexes and an analysis of local cohomology conditions. The fact that  $\text{Sel}(F, \mathcal{T})$  is often torsion follows from recent results of S.Zhang and Y.Tian in [YZZ08].

## 1.1 Notations

Let  $F$  be a totally real field and let  $p \geq 5$  be a rational prime. Let  $\delta_{F,p}$  be the defect of Leopoldt's conjecture for  $F$  at  $p$ , hence the rank of the maximal  $\mathbb{Z}_p$ -extension of  $F$  minus 1. For  $v$  a finite place of a finite extension of  $\mathbb{Q}$ , let  $\text{Fr}(v)$  be the geometric Frobenius morphism. Let  $\mathbb{A}, \mathbb{A}_f, \mathbb{A}_F$  and  $\widehat{F}$  be respectively the adèles and finite adèles of  $\mathbb{Q}$  and the adèles and finite adèles of  $F$ . Let  $I = \{\tau_i\}_{i=1, \dots, d}$  be the set of infinite places of  $F$ . If  $L$  is a finite extension of  $\mathbb{Q}$  or  $\mathbb{Q}_\ell$ , let  $G_L$  be the absolute Galois group of  $L$ . For  $R$  a complete local noetherian ring containing  $\mathbb{Z}_p$ , a  $G_L$ -representation  $(T, \rho, R)$  is an  $R$ -module free of finite rank endowed with a continuous action of  $G_L$  given by the representation morphism  $\rho$ . Let  $S$  be a complete local noetherian ring containing  $\mathbb{Z}_p$ . An  $S'$ -specialization  $\kappa$  of  $S$  is a morphism of  $\mathbb{Z}_p$ -algebras from  $R$  to  $S$ . The  $S$ -specialization  $T_\kappa$  of  $T$  attached to  $\kappa$  is the  $G_F$ -representation  $(T \otimes_{R, \kappa} S, \rho, S)$  with trivial  $G_F$ -action on  $S$  and  $G_F$ -action through  $\rho$  on  $T$ . A sub-specialization of  $T_\kappa$  is an  $S'$ -specialization  $T_{\kappa'}$  of  $T$  such that  $T_\kappa$  is an  $S$ -specialization of  $T_{\kappa'}$ .

## 2 Selmer structures of nearly ordinary Hecke algebra

### 2.1 Generalities on nearly ordinary Hecke algebras

#### 2.1.1 Representations of $\mathbf{G}$ and Hecke operators

**2.1.1.1 Hilbert modular forms** Let  $\mathbf{G}$  be the reductive algebraic group  $\text{GL}_2$  over  $F$ . We identify  $\mathbf{G}(\mathbb{A}_F)$  with the restricted direct product of the groups  $\mathbf{G}(F_v)$  with respect to  $\mathbf{G}(\mathcal{O}_{F,v})$ . To an ideal  $\mathcal{N}$  of  $\mathcal{O}_F$ , we attach the standard compact open subgroups  $U_0, U_1, U^1$  and  $U_{1,1}$  of  $\mathbf{G}(\mathbb{A}_f)$ . These groups are maximal outside  $\mathcal{N}$  and verify

$$U_0(\mathcal{N})_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{GL}_2(\mathcal{O}_{F,v}) \mid c \equiv 0 \pmod{\mathcal{N}} \right\} \quad (2.1.1)$$

$$U_1(\mathcal{N})_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathcal{N})_v \mid d \equiv 1 \pmod{\mathcal{N}} \right\} \quad (2.1.2)$$

$$U^1(\mathcal{N})_v = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(\mathcal{N})_v \mid a \equiv 1 \pmod{\mathcal{N}} \right\} \quad (2.1.3)$$

$$U_{1,1}(\mathcal{N})_v = U_1(\mathcal{N})_v \cap U^1(\mathcal{N})_v \quad (2.1.4)$$

at  $v|\mathcal{N}$ . Let  $U$  be a compact open subgroup of  $G(\mathbb{A}_F)$  verifying

$$U = \prod_{v|p} \mathrm{GL}_2(\mathcal{O}_{F,v}) \prod_{v \nmid p\infty} U_v, \quad U_v \subset \mathrm{GL}_2(\mathcal{O}_{F,v})$$

and such that there exists an  $\mathcal{N}$  with  $U_{1,1}(\mathcal{N}) \subset U \subset U_0(\mathcal{N})$ . For  $s \geq 1$ , define  $U_0(s), U_1(s), U^1(s)$  and  $U_{1,1}(s)$  by

$$U_*(s) = U \cap U_*(p^s)$$

for  $* =_{0,1}, ^1,_{1,1}$ . A weight  $k$  is an element of  $\mathbb{Z}[I]$ ; an arithmetic weight is a weight such that  $k_v \geq 2$  for all  $v \in I$  and such that the  $k_v$  have constant parity; a parallel weight is an integral multiple of  $t$  the weight such that  $t_v = 1$  for all  $v \in I$ ; a parallel defect is a weight  $\nu$  such that  $\nu_v$  is non-negative for all  $v$  and such that one  $\nu_v$  at least is zero. To an arithmetic weight are associated  $(m, \nu, \mu) \in \mathbb{Z}[I]^2 \times \mathbb{Z}$  as follows:

$$m = k - 2t \tag{2.1.5}$$

The weight  $\nu$  is a parallel defect such that  $\mu \cdot t = m + 2\nu$  is a parallel weight. For  $k \in \mathbb{Z}[I]$  and  $x \in \mathbb{C}^I$ , let  $x^k \in \mathbb{C}$  be  $\prod_{v|\infty} x_v^{k_v}$ . We refer to [Shi78, Section 1], [SW99, Section 3.1] or to [Nek06,

Section 12.3] for definitions of holomorphic cuspforms. For  $k$  an arithmetic weight, let  $S_k(U)$  and  $S_k$  be respectively the  $\mathbb{C}$ -vector space of holomorphic cuspforms of weight  $k$  and level  $U$  and the direct limit  $\varinjlim S_k(U)$ . Then  $S_k$  is an admissible representation of  $\mathbf{G}(\widehat{F})/\widehat{F}^\times$  under the right action

of  $\mathbf{G}(\widehat{F})$  on  $f \in S_k$ . This representation decomposes into a direct sum of irreducible admissible representations  $V_\pi$ . Then  $S_k(U)$  is the direct sum of the  $V_\pi$  such that  $V_\pi^U \neq 0$ . To  $f \in S_k(U)$  is thus attached an automorphic representation  $\pi(f)$  of  $\mathbf{G}(\mathbb{A}_F)$ . We recall that for  $v$  a finite place of  $F$ , the  $\mathbf{G}(F_v)$ -representation  $\pi(f)_v$  is either an irreducible principal series, a twisted Steinberg or a supercuspidal representation.

**2.1.1.2 Hecke operators** For  $g \in \mathbf{G}(\widehat{F})$ , the correspondence  $[UgU]$  defines an endomorphism of  $S_k(U)$  by:

$$[UgU] \cdot f(x) = \sum_{g_i} f(xg_i^{-1}), \quad [UgU] = \prod_{g_i} Ug_i$$

This in particular defines an action of  $\mathbf{G}(s) = U_0(s) \cdot \mathcal{O}_F^\times / U_{1,1}(s) \cdot \mathcal{O}_F^\times$  in the following way. For

$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in U_0(s)$  and  $y \in \mathcal{O}_F^\times$ , let  $x \cdot y$  act via the correspondence  $\prod_{v|p} \omega(a_v)^{-\mu} [U_{1,1}gU_{1,1}]$ . For

$v \notin S_B$ , we choose an element  $\lambda^{(v)} \in \mathcal{O}_F \otimes \widehat{\mathbb{Z}}$  such that  $\lambda_v^{(v)}$  is a uniformizer of  $\mathcal{O}_{F,v}$  and  $\lambda_v^{(v)}$  is equal to 1 outside  $v$ . If  $v|p$ , we require furthermore that  $\lambda_v^{(v)}$  belongs to  $\mathcal{O}_F$  and that  $\lambda_v^{(v)}$  belongs to  $\mathcal{O}_{F,v'}^\times$  for  $v'|p$  and  $v' \neq v$ . Let  $T(v)$  and  $S(v)$  be the following correspondences:

$$T(v) = \left[ U \begin{pmatrix} 1 & 0 \\ 0 & \lambda^{(v)} \end{pmatrix} U \right], \quad S(v) = \left[ U \begin{pmatrix} \lambda^{(v)} & 0 \\ 0 & \lambda^{(v)} \end{pmatrix} U \right]$$

The full abstract Hecke algebra is the  $\mathbb{Z}$ -algebra generated by the correspondence  $g \cdot y$  of  $\mathbf{G}(s)$ , and by the  $T(v)$  and the  $S(v)$  for all  $v \notin S_B$ . Let  $k$  be an arithmetic weight. For  $v|p$ , let  $T_0(v)$  be  $(\lambda_v^{(v)})^{-\nu} T(v)$ . Let  $\tilde{p} \in \mathbb{A}_F$  be  $p$  at  $v|p$  and 1 at  $v \notin p$  and let  $T_0(p)$  be

$$p^{-\nu} \left[ U \begin{pmatrix} 1 & 0 \\ 0 & \tilde{p} \end{pmatrix} U \right]$$

The Hecke algebra  $\mathfrak{h}_k(U)$  is the sub-algebra of  $\text{End}(S_k(U))$  generated by the image of the action of  $\mathbf{G}(s)$ , by the images of the  $T_0(v)$ , of  $T(p)$  and of  $T(v)$  and  $S(v)$  for all  $v \notin S_B$  such that  $U_v$  is maximal. As  $S_k(U) = \bigoplus V_\pi^U$ , the Hecke algebra acts on each  $V_\pi^U$

**2.1.1.3 Nearly ordinary representations** Let  $f$  be a newform in  $S_k(U_{1,1}(s))$  and  $\pi(f)$  be the attached automorphic representation. Then  $f$  or  $\pi(f)$  are said to be nearly ordinary at  $v|p$  if there exists a  $v$ -good line on  $V_{\pi(f)_v}^{U_{1,1}(s)_v}$ , this is to say a line on which  $T_0(v)$  acts via a unit in the ring of integers of  $\bar{\mathbb{Q}}_p$ . In that case,  $\pi(f)_v$  is either a principal series representation  $\pi(\eta_v | \cdot |v^{-1/2}, \xi_v | \cdot |v^{-1/2})$  or a special representation  $\pi(\xi_v | \cdot |^{-1/2}, \xi_v | \cdot |^{1/2})$  with  $\lambda_v^{-\nu} \xi_v(\lambda_v)$  a unit in the ring of integers of  $\bar{\mathbb{Q}}_p$ . They are said to be ordinary at  $v|p$  if moreover  $\xi_v$  is unramified. They are said to be (nearly) ordinary if they are (nearly) ordinary at all  $v|p$ . We recall that the nearly ordinary subspace  $S_k^{\text{ord}}(U)$  of  $S_k(U)$  is the span in  $\bigoplus V_\pi^U$  of well-chosen  $w(\pi)$  which are eigenvectors under the action of the Hecke algebra and which span  $v$ -good lines for all  $v$  dividing  $p$ .

## 2.1.2 The nearly ordinary Hecke algebra

**2.1.2.1 The weight Iwasawa algebra** The constructions of 2.1.1 make sense when  $s$  goes to infinity, or equivalently as  $U$  shrinks at  $v|p$ . Let  $\mathbf{G}(\infty)$  be the inverse limit on  $s$  of the  $\mathbf{G}(s)$ , with transition maps induced by the inclusion of  $U_0(s+1)$  in  $U_0(s)$ . Define:

$$Z(s) = (U_0(s) \cap \widehat{F}^\times) \cdot \mathcal{O}_F^\times / (U_{1,1}(s) \cap \widehat{F}^\times) \cdot \mathcal{O}_F^\times, \quad Z(\infty) = \varprojlim_s Z(s)$$

Then  $Z(\infty)$  is a  $\mathbb{Z}_p$ -module of rank  $1 + \delta_{F,p}$ . Moreover:

$$\mathbf{G}(s) \xrightarrow{\sim} (\mathcal{O}_F/p^s)^\times \times Z(s), \quad \mathbf{G}(\infty) \xrightarrow{\sim} (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times Z(\infty)$$

This identification defines correspondences  $T_y = (y, 1)$  and  $S_x = (1, x)$  for  $(y, x) \in (\mathcal{O}_F \otimes \mathbb{Z}_p)^\times \times Z(\infty)$ . Thus  $\mathbf{G}(\infty)$  is a  $\mathbb{Z}_p$ -module of rank  $1 + \delta_{F,p} + d$ . Let  $\Lambda_{\mathcal{O}}$  be the regular ring of dimension  $2 + \delta_{F,p} + d$  equal to  $\mathcal{O}[[X_1, \dots, X_{1+\delta_{F,p}}, Y_1, \dots, Y_d]]$ . A choice of a system of generators of the maximal  $\mathbb{Z}_p$ -free summand of  $Z(\infty)$  and of  $(\mathcal{O}_F \otimes \mathbb{Z}_p)^\times$  provides an isomorphism between  $\Lambda_{\mathcal{O}}$  and  $\mathcal{O}[[\mathbf{G}(\infty)/\mathbf{G}(\infty)_{\text{tors}}]]$  given by  $X_i \rightarrow S_{x_i}$  and  $Y_i \rightarrow T_{y_i}$ . We fix such a choice and henceforth identify  $\Lambda_{\mathcal{O}}$  with  $\mathcal{O}[[\mathbf{G}(\infty)/\mathbf{G}(\infty)_{\text{tors}}]]$ . For  $\mathcal{O}'$  a discrete valuation ring flat over  $\mathbb{Z}_p$ , let  $\Lambda_{\mathcal{O}'}$  be  $\Lambda_{\mathcal{O}} \otimes_{\mathcal{O}} \mathcal{O}'$ .

An  $\mathcal{O}$ -algebra morphism  $\lambda$  of  $\Lambda_{\mathcal{O}}$  with values in  $\bar{\mathbb{Q}}_p$  is called an arithmetic point if there exists a finite character  $\psi$  of  $Z(\infty)$ , a non-negative integer  $\mu$  and a parallel defect  $\nu$  such that  $\lambda$  is of the following form:

$$\begin{aligned} \lambda: \quad \mathbf{G}(\infty) &\longrightarrow \bar{\mathbb{Q}}_p \\ (a, z) &\longmapsto \psi \chi_{\text{cyc}}^\mu(z) a^\nu \end{aligned}$$

The arithmetic weight  $(\mu + 2) \cdot t - 2\nu$  and the character  $\psi$  are called the weight and the character of  $\lambda$ . The set of arithmetic primes  $\text{Spec}^{\text{arith}}(\Lambda_{\mathcal{O}}) \subset \text{Spec}(\Lambda_{\mathcal{O}})$  of  $\Lambda_{\mathcal{O}}$  is the set of primes equal to the kernel of an arithmetic point. If  $R$  is a finite  $\Lambda_{\mathcal{O}}$ -algebra, the set of arithmetic primes  $\text{Spec}^{\text{arith}}(R)$  of  $R$  is the set of primes  $\mathcal{P}$  verifying  $\mathcal{P} \cap \Lambda_{\mathcal{O}} \in \text{Spec}^{\text{arith}}(\Lambda_{\mathcal{O}})$ . We will often make use of the elementary observation that arithmetic primes do not contain elements of  $\bar{\mathbb{Q}}_p$ .

**2.1.2.2 Nearly ordinary Hecke algebra** The nearly ordinary Hecke algebra  $\mathbf{T}_k(U(s), \mathcal{O})$  of weight  $k$  and level  $s$  is the sub-algebra of  $\text{End}_{\mathbb{C}}(S_k^{\text{ord}}(U(s)))$  generated over  $\mathcal{O}$  by the same

operators as the Hecke algebra. The  $\mathcal{O}$ -algebra  $\mathbf{T}_k(U(s), \mathcal{O})$  is finite flat and reduced. Let the nearly ordinary Hecke algebra  $\mathbf{T}_\infty(U, \mathcal{O})$  be the inverse limit on  $s$  of the  $\mathbf{T}_2(U(s), \mathcal{O})$ . According to [Hid88, Theorem 1], the  $\mathcal{O}$ -algebra  $\mathbf{T}_\infty(U, \mathcal{O})$  is also isomorphic to the inverse limit  $\varprojlim_s \mathbf{T}_k(U(s), \mathcal{O})$

for any arithmetic weight  $k$ . The nearly ordinary Hecke algebra is a finite torsion-free  $\Lambda_{\mathcal{O}}$ -module, hence a semi-local ring. When  $\mathcal{O}$  and  $U$  are clear from the context or unimportant, we write  $\Lambda$  for  $\Lambda_{\mathcal{O}}$  and  $\mathbf{T}_\infty$  for  $\mathbf{T}_\infty(U, \mathcal{O})$ .

The nearly ordinary Hecke algebra is a finite  $\Lambda_{\mathcal{O}}$ -algebra by the inclusion of  $\mathbf{G}(\infty)$  in  $\mathbf{T}_\infty(U, \mathcal{O})$ . If  $f$  is a nearly ordinary eigenform under the action of  $\mathbf{T}_\infty(U, \mathcal{O})$ , then the map  $\lambda_f$  sending  $T \in \mathbf{T}_\infty(U, \mathcal{O})$  to the eigenvalue of  $f$  under  $T$  is an arithmetic point of  $\mathbf{T}_\infty(U, \mathcal{O})$ . Conversely, if  $\lambda$  is an arithmetic point of  $\mathbf{T}_\infty(U, \mathcal{O})$  of weight  $k$ , then there exists a nearly ordinary eigenform  $f$  such that  $\lambda(T(v))$  and  $\lambda(S(v))$  are equal to the eigenvalues of  $T(v)$  and  $S(v)$  acting on  $f$  for all  $v \notin S_B$  such that  $U_v$  is maximal. To  $f$  a nearly ordinary eigenform are thus attached an arithmetic prime  $\mathcal{P}_f$ , a maximal ideal  $\mathfrak{m}$ , a local ring  $\mathbf{T}_{\mathfrak{m}} = \mathbf{T}_\infty(U, \mathcal{O})_{\mathfrak{m}}$  and a unique minimal prime  $\mathfrak{a} \subset \mathcal{P}_f$  such that  $\lambda_f$  factors through the local domain  $R = \mathbf{T}_{\mathfrak{m}}/\mathfrak{a}$ .

**Lemma 2.1.** *Let  $\mathcal{P}$  be a minimal prime of  $\text{Spec } \mathbf{T}_\infty$ . Then the set of arithmetic primes of fixed weight containing  $\mathcal{P}$  is Zariski-dense in  $\text{Spec } \mathbf{T}_\infty/\mathcal{P}$ .*

**Lemma 2.2.** *Let  $\mathcal{P} \in \text{Spec}^{\text{arith}} \mathbf{T}_\infty$  be an arithmetic prime over  $\mathfrak{p} \in \text{Spec}^{\text{arith}} \Lambda$ . Then  $(\mathbf{T}_\infty)_{\mathcal{P}}$  is an étale extension of  $\Lambda_{\mathfrak{p}}$ .*

**Lemma 2.3.** *Let  $f$  be a nearly ordinary eigenform of weight  $k$  and level  $U$ . For all prime numbers  $p$  such that  $f$  is  $p$ -ordinary, there exists a local factor  $\mathbf{T}_{\mathfrak{m}_p}$  attached to  $f$ . The rings  $\mathbf{T}_{\mathfrak{m}_p}$  are almost all regular local rings.*

*Proof.* By finiteness of  $S_k(U)$  over  $\mathbb{Z}$ , for  $p$  large enough, the arithmetic point  $\lambda_f$  is not congruent modulo  $p$  to another arithmetic point of the same weight and level. For such a  $p$ , let  $\mathfrak{m}$  be the maximal ideal attached to  $f$ , let  $\mathcal{P}$  be the kernel of  $\lambda_f$  and let  $\mathfrak{p}$  be  $\mathcal{P} \cap \Lambda_{\mathcal{O}}$ . After a finite flat extension of  $\mathcal{O}$  if necessary,  $\mathbf{T}_k(U, \mathcal{O})_{\mathfrak{m}}$  is then isomorphic to  $\mathcal{O}$  and the local ring  $\mathbf{T}_{\mathfrak{m}}/\mathfrak{p}$  injects in  $\mathbf{T}_k(U, \mathcal{O})_{\mathfrak{m}}$  so is itself a discrete valuation ring. The prime ideal  $\mathfrak{p}$  being generated by a regular sequence contained in a system of parameters,  $\mathfrak{m}$  is generated by  $(\mathfrak{p}, \pi_{\mathcal{O}})$  and so  $\mathbf{T}_{\mathfrak{m}}$  is a regular ring.  $\square$

The following lemma is often called the control theorem for ordinary Hecke algebras.

**Lemma 2.4.** *Let  $\mathfrak{m}$  be the maximal ideal attached to a nearly ordinary eigenform. Assume that  $R = \mathbf{T}_{\mathfrak{m}}/\mathfrak{a}$  is regular and let  $\mathcal{P} \in \text{Spec}^{\text{arith}} R$  be an arithmetic prime of weight  $k$  and level  $U$ . Then  $R/\mathcal{P}$  is isomorphic to  $\mathbf{T}_k^{\text{ord}}(U, \mathcal{O}')_{\mathfrak{m}}/\mathfrak{a}$ .*

## 2.1.3 Galois representations

**2.1.3.1 Galois representations of nearly ordinary eigenforms** Let  $f \in S_k^{\text{ord}}(U, \varphi)$  be a nearly ordinary eigenform under the action of  $\mathbf{T}_k(U)$ . Let  $\mathcal{N}$  be the conductor of  $\pi(f)$ . Let  $L_{\mathfrak{p}}$  be a finite extension of  $\mathbb{Q}_p$  containing the Hecke eigenvalues of  $f$ , let  $\mathcal{O}'$  be its ring of integers and  $\mathbb{F}$  be its residual field. Let  $\lambda_f$  be the arithmetic point of  $\mathbf{T}_\infty(U, \mathcal{O}')$  induced by  $f$ . There exists a continuous, irreducible  $G_F$ -representation  $(V(f), \rho_f, L_{\mathfrak{p}})$  free of rank 2 which is unramified outside  $\mathcal{N}p$  and which verifies

$$\det(1 - \text{Fr}(v)X | V(f)) = 1 - \lambda_f(T(v))X + \lambda_f(S(v))N_{F/\mathbb{Q}}vX^2 \quad (2.1.6)$$

for all  $v \nmid \mathcal{N}p$ . If  $\tau$  is a complex conjugation, then  $\rho_f(\tau)$  is odd. Hence, the  $G_F$ -representation  $V(f)$  is absolutely irreducible and thus characterized up to isomorphism by (2.1.6). If  $v|p$ , the local

$G_{F_v}$ -representation  $V(f)$  admits a one dimensional sub-representation  $V(f)_v^+$  on which  $\lambda_v^{(v)}$  acts as  $\lambda_f(T_0(v))$  and  $y \in \mathcal{O}_{F,v}^\times$  acts as  $\lambda_f(T_y)$  (after identification of  $G_{F_v}^{ab}$  with  $\widehat{F}^\times$ ). Hence,  $V(f)$  fits in an exact sequence of  $G_{F_v}$ -representations

$$0 \longrightarrow V(f)_v^+ \longrightarrow V(f) \longrightarrow V(f)_v^- \longrightarrow 0$$

with  $V(f)_v^+$  unramified if  $k$  is parallel. If  $v|\mathcal{N}$ , then  $I_v$  acts on  $V(f)$  non-trivially and through an infinite quotient if and only if  $\pi(f)_v$  is a Steinberg representation. If it acts through a finite quotient, it does so reducibly if and only if  $\pi(f)_v$  is a principal series. In particular, the automorphic type of  $\pi(f)$  does not change after twisting by a character. The eigenvalues  $\alpha_i$  of a lift of  $\text{Fr}(v)$  to  $G_{F_v}$  seen as complex numbers have modules  $(N_{F/\mathbb{Q}}v)^{w/2}$  with  $w = k - 1$  if  $\pi(f)_v$  is either principal series or supercuspidal and  $w \in \{k - 2, k\}$  if  $\pi(f)_v$  is Steinberg.

Remark that the trace of  $\rho_f$  determines a pseudo-representation  $\tilde{\rho}_f$  with values in  $\mathcal{O}'$  hence a semi-simple residual representation  $(\overline{T}, \tilde{\rho}_f, \mathbb{F})$ . If  $\tilde{\rho}_f$  is absolutely irreducible, there thus exists a  $G_F$ -representation  $(T(f), \rho_f)_{\mathcal{O}'}$  verifying the same properties as  $V(f)$  by [Nys96, Théorème 1].

**2.1.3.2 Galois representations attached to minimal ideals of  $\mathbf{T}_\infty$**  Let  $\mathcal{P}_{\min}$  be a minimal prime of  $\mathbf{T}_\infty$ , let  $R^{\min}$  be the local domain  $\mathbf{T}_\infty/\mathcal{P}_{\min}$  and let  $\lambda_m$  be the projection from  $\mathbf{T}_\infty$  to  $R^{\min}$ . The pseudo-representations attached to arithmetic primes containing  $\mathcal{P}_{\min}$  define by lemma 2.1 and [Wil88, Lemma 2.2.3] a pseudo-representation  $\tilde{\rho}$  with values in  $R^{\min}$ . Thus, there exists a continuous, absolutely irreducible, semi-simple  $G_F$ -representation  $(\mathcal{V}, \rho_m, \text{Frac}(R^{\min}))$  which is unramified outside  $\mathcal{N}p$  and which verifies

$$\det(1 - \text{Fr}(v)X | \mathcal{V}) = 1 - \lambda_m(T(v))X + \lambda_m(S(v))N_{F/\mathbb{Q}}vX^2 \quad (2.1.7)$$

for all  $v \nmid \mathcal{N}p$ . If  $v|p$ , the local  $G_{F_v}$ -representation  $\mathcal{V}$  admits a one dimensional sub-representation  $\mathcal{V}_v^+$  on which  $\lambda_v^{(v)}$  acts as  $\lambda_m(T_0(v))$  and  $y \in \mathcal{O}_{F,v}^\times$  acts as  $\lambda_m(T_y)$ . Hence,  $\mathcal{V}$  fits in an exact sequence of  $G_{F_v}$ -representations:

$$0 \longrightarrow \mathcal{V}_v^+ \longrightarrow \mathcal{V} \longrightarrow \mathcal{V}_v^- \longrightarrow 0$$

Let  $\mathcal{P}_f$  be an arithmetic prime containing  $\mathcal{P}_{\min}$  attached to a nearly ordinary newform  $f$ , let  $\mathcal{P} \in \text{Spec}(R^{\min})$  be  $\mathcal{P}_f/\mathcal{P}_{\min}$  and let  $\lambda_{\mathcal{P}}$  be the morphism  $\mathbf{T}_\infty \longrightarrow R_{\mathcal{P}}^{\min}$ . The ring  $R_{\mathcal{P}}^{\min}$  is normal by lemma 2.2 so  $(\mathcal{V}, \rho_{\min}, \text{Frac}(R^{\min}))$  descends to a  $G_F$ -representation  $(\mathcal{T}_{\mathcal{P}}, \rho_{\mathcal{P}}, R_{\mathcal{P}}^{\min})$  whose representation space we also sometimes denote by  $\mathcal{I}_f$ . Property (2.1.7) and absolute irreducibility show that  $\mathcal{T}_{\mathcal{P}}/\mathcal{P}$  is isomorphic to  $\rho_f$ . The representation  $(\mathcal{T}_{\mathcal{P}}, \rho_{\mathcal{P}}, R_{\mathcal{P}}^{\min})$  verifies the same properties as  $\mathcal{V}$  with  $\lambda_m$  replaced by  $\lambda_{\mathcal{P}}$ .

**2.1.3.3 Galois representations attached to maximal ideals of  $\mathbf{T}_\infty$**  Let  $\mathfrak{m}$  be a maximal ideal of  $\mathbf{T}_\infty$  and  $\lambda_{\mathfrak{m}}$  be the natural morphism  $\mathbf{T}_\infty \longrightarrow \mathbf{T}_{\mathfrak{m}}$ . Let  $\mathfrak{a}$  be a minimal prime contained in  $\mathfrak{m}$  and let  $R$  be  $\mathbf{T}_{\mathfrak{m}}/\mathfrak{a}$ . Let  $\tilde{\rho}_{\mathfrak{m}}$  be the residual representation  $\tilde{\rho}_f$  attached to any arithmetic prime contained in  $\mathfrak{m}$ . As  $\mathbf{T}_{\mathfrak{m}}$  is a factor of  $\mathbf{T}_\infty$ , the pseudo-representations attached to the minimal primes contained in  $\mathfrak{m}$  define a pseudo-representation  $\tilde{\rho}_{\mathfrak{m}}$  with values in  $\mathbf{T}_{\mathfrak{m}}/\cap \mathcal{P}_m$ , hence in  $\mathbf{T}_{\mathfrak{m}}$  as this ring is reduced. If the residual representation  $\tilde{\rho}_{\mathfrak{m}}$  is irreducible, then [Nys96, Théorème 1] implies that  $\tilde{\rho}_{\mathfrak{m}}$  is the trace of a  $G_F$ -representation  $(\mathcal{T}_{\mathfrak{m}}, \rho_{\mathfrak{m}}, \mathbf{T}_{\mathfrak{m}})$ . Reducing modulo  $\mathfrak{a}$  gives a  $G_F$ -representation  $(\mathcal{T}(\mathfrak{a}), \rho_{\mathfrak{m}}, R)$ . If  $\mathcal{T}(\mathfrak{a})$  exists, it verifies the same properties as  $\mathcal{V}$  with  $\lambda_m$  replaced by  $\lambda_{\mathfrak{m}} \bmod \mathfrak{a}$ .

**2.1.3.4 Rigidity of automorphic types** The automorphic types of arithmetic points in  $\mathcal{V}$  and  $\mathcal{T}(\mathfrak{a})$  are rigid in the following sense.

**Lemma 2.5.** *Let  $v \nmid p$  be a finite place. Then the arithmetic specializations of  $\mathcal{V}$  and  $\mathcal{T}(\mathfrak{a})$  have all the same automorphic type at  $v$ .*

*Proof.* Let  $I_v^w$  and  $I_v^t$  be respectively the wild and tame inertia subgroup. The representation  $\mathcal{V}$  is continuous so admits an  $R^{\min}$ -lattice  $T$  stable under  $G_F$ . The residue field of  $R^{\min}$  is finite of characteristic  $p \neq N_{F/\mathbb{Q}}v$  so there exists an open normal subgroup  $U$  of  $I_v$  such that  $\rho|_U$  is unipotent. Let  $\tilde{\rho}$  be the pseudo-representation with values in  $R^{\min}$  attached to  $\mathcal{V}$  and let  $f$  and  $g$  be two arithmetic specializations of  $T$  with associated arithmetic primes  $\mathcal{P}_f$  and  $\mathcal{P}_g$ .

Assume first that the action of  $U$  on  $\mathcal{V}$  is trivial, so that  $\tilde{\rho}(I_v)$  is a finite subgroup of  $\bar{\mathbb{Q}}_p$ . Then it is also trivial on  $V(f)$  and  $V(g)$  so  $\pi(f)_v$  and  $\pi(g)_v$  are not Steinberg representations. The two pseudo-representations  $\tilde{\rho}_f$  and  $\tilde{\rho}_g$  of  $I_v$  with coefficients in  $\bar{\mathbb{Q}}_p$  verify  $\tilde{\rho}_f = \tilde{\rho} \bmod \mathcal{P}_f = \tilde{\rho}_g$ . The  $\bar{\mathbb{Q}}_p[I_v]$ -modules  $V(f)$  and  $V(g)$  are thus isomorphic. In particular, they are both reducible or both irreducible. Hence, the automorphic representations  $\pi(f)_v$  for arithmetic points of  $R$  are either all principal series or all supercuspidal.

Assume now that there then exists an arithmetic point  $f$  such that  $I_v$  acts through an infinite quotient on  $V(f)$  and thus such that  $\pi(f)_v$  is Steinberg. Then  $U$  acts non-trivially. Twisting by a finite order character if necessary, we can assume that  $\pi(f)_v$  is an unramified Steinberg representation. Then  $V(f)^{I_v}$  is of dimension 1 and  $I_v^w$  acts trivially on  $V(f)$ . As  $I_v^w$  acts through a finite quotient on  $\mathcal{I}_f$ , its action on  $\mathcal{I}_f$  is also trivial. Thus, there exists an open subgroup  $\langle \sigma \rangle$  of the tame inertia group acting unipotently and non-trivially on  $\mathcal{I}_f$ , which implies that  $\mathcal{I}_f^{\sigma=1}$  is free of rank 1 over  $R_{\mathcal{P}}$ . As  $V(f)^{\sigma=1}$  and  $\mathcal{I}_f^{\sigma=1}$  are both of rank 1,  $V^{\sigma=1}$  is of rank 1 for any subspecialization  $V$  of  $V(f)$ . Let  $(x_i)_{1 \leq i \leq r}$  be a regular sequence generating the ideal  $\mathcal{P}_f$ . Computing ranks shows that the short sequence

$$0 \longrightarrow \mathcal{I}_f^{\sigma=1}/(x_1, \dots, x_n) \xrightarrow{x_{n+1}} \mathcal{I}_f^{\sigma=1}/(x_1, \dots, x_n) \longrightarrow (\mathcal{I}_f/(x_1, \dots, x_{n+1}))^{\sigma=1} \longrightarrow 0$$

is exact for all  $n \geq 0$ . The group  $I_v^t/\langle \sigma \rangle$  is finite so taking  $I_v^t/\langle \sigma \rangle$ -invariants respect exact sequences. Hence,  $\mathcal{I}_f^{I_v}$  is  $R_{\mathcal{P}}$ -free of rank 1. Thus  $\mathcal{V}^{I_v}$  is  $\text{Frac}(R^{\min})$ -free of rank 1. The determinant  $\det \mathcal{V}$  of  $\mathcal{V}$  restricted to  $I_v$  is a finite order character by the equivalent of (2.1.6) for  $\mathcal{V}$  so is equal to  $\det V(f)$  and thus trivial. The  $\text{Frac}(R^{\min})[\langle \text{Fr}(v) \rangle]$ -modules  $\mathcal{V}(-1)^{I_v}$  and  $\mathcal{V}/\mathcal{V}^{I_v}$  are thus isomorphic by the monodromy operator. Let  $\chi$  be the character through which  $\text{Fr}(v)$  acts on  $\mathcal{V}(-1)^{I_v}$ , and thus also on  $\mathcal{V}/\mathcal{V}^{I_v}$ . Then  $\det \mathcal{V}(\text{Fr}(v))$  is equal to  $\chi^2(\text{Fr}(v))\chi_{\text{cyc}}^{-1}(\text{Fr}(v))$  so  $\chi^2 = 1$ . Hence  $\chi^2$  is of finite order and so  $\pi(f)_v = \text{St}(\chi)$ . By specialization,  $\pi(g)_v = \text{St}(\chi)$  for  $g$  an arithmetic specialization.

The statement for  $\mathcal{T}(\mathfrak{a})$  comes from the fact that  $\mathcal{T}(\mathfrak{a})$  localized at an arithmetic prime  $\mathcal{P}_f$  is isomorphic to  $\mathcal{I}_f$ .  $\square$

**2.1.3.5 Analogies between  $\mathcal{I}_f$  and  $\mathcal{T}_{\mathfrak{m}}$**  When  $f$  is such that  $R = \mathbf{T}_{\mathfrak{m}}/\mathfrak{a}$  is a regular local ring, the  $G_F$ -representations  $\mathcal{I}_f$  and  $\mathcal{T}(\mathfrak{a})$  share many formal properties: they are both free modules of rank 2 over a regular ring satisfying a control theorem and their residual representations are absolutely irreducible.

## 2.2 Selmer structures

### 2.2.1 Generalities

We keep the notation of section 2.1.3 and assume that we have chosen a nearly ordinary eigenform  $f$ , and thus a maximal ideal  $\mathfrak{m}$  and an arithmetic prime  $\mathcal{P}_f$ . Let  $\mathbb{F}$  be the residue field of  $\mathbf{T}_{\mathfrak{m}}$ . Let  $S$  be a principal artinian ring finite over  $\mathbb{F}$ , or a finite flat extension of  $\mathcal{O}$ , or a finite extension of  $L_{\mathfrak{p}}$ , or  $\text{Frac}(R^{\min})$  or a non-zero integral quotient of  $R_{\mathcal{P}}^{\min}$  or  $R$ . The constructions of 2.1.3 provide by specialization a pseudo-representation  $\tilde{\rho}$  with coefficients in  $S$ . Assuming that  $\tilde{\rho}_{\mathfrak{m}}$  is irreducible

in the first two and the last cases, we thus have a  $G_F$ -representation  $(T, \rho, S)$  with coefficients in  $S$ . We say that  $T$  contains an arithmetic point if  $T$  admits an arithmetic specialization.

Let  $L$  be a finite extension of  $F$ . Let  $\Sigma_p$  be the set of places of  $L$  above  $p$  and let  $\Sigma$  be a finite set of places of  $L$  containing  $\Sigma_p$ , the places of ramification of  $T$  and the infinite places. Let  $L_\Sigma$  be the maximal Galois extension of  $L$  unramified outside  $\Sigma$ . Let  $G$  be  $\text{Gal}(L_\Sigma/L)$  or  $G_{L_v}$  or  $I_v$  for  $v$  a place of  $L$ . Then the continuous cohomology groups  $H^i(G, T)$  are finitely generated  $S$ -modules by [Nek06, Proposition 4.2.3]. Let  $C_{\text{cont}}^\bullet(G, T)$  denote the complex of continuous cochains with values in  $T$  and  $\text{R}\Gamma(G, T)$  the corresponding object in the derived category. We write  $H^i(\text{Gal}(L_\Sigma/L), T)$  and  $H^i(G_{L_v}, T)$  respectively for  $H^i(L, T)$  and  $H^i(L_v, T)$  and similarly with complexes.

**Lemma 2.6.** *Let  $\mathbf{x} = (x_1, \dots, x_r)$  be an  $S$ -regular sequence. For  $i \geq 1$ , let  $\mathbf{x}_i$  be  $(x_1, \dots, x_i)$ . Then:*

$$\text{R}\Gamma(G, T) \otimes_{S/\mathbf{x}_i}^{\text{L}} \xrightarrow{\sim} \text{R}\Gamma(G, T/\mathbf{x}_i T)$$

*Proof.* The exact sequence

$$0 \longrightarrow S/\mathbf{x}_i \xrightarrow{x_{i+1}} S/\mathbf{x}_i \longrightarrow S_{\mathbf{x}_{i+1}} \longrightarrow 0$$

induces an exact sequence

$$0 \longrightarrow T/\mathbf{x}_i \xrightarrow{x_{i+1}} T/\mathbf{x}_i \longrightarrow T_{\mathbf{x}_{i+1}} \longrightarrow 0$$

hence an exact sequences of complexes:

$$0 \longrightarrow C_{\text{cont}}^\bullet(G, T/\mathbf{x}_i) \longrightarrow C_{\text{cont}}^\bullet(G, T/\mathbf{x}_i) \longrightarrow C_{\text{cont}}^\bullet(G, T_{\mathbf{x}_{i+1}}) \longrightarrow 0$$

□

We sum up the previous property by saying that  $\text{R}\Gamma(G, \cdot)$  descends perfectly. More generally, we say that a functor  $X(\cdot)$  from the category of free modules over quotient of a ring  $S$  to the category of bounded complexes of such modules of finite type descends  $\mathbf{x}$ -perfectly if:

$$X(T) \otimes_{S/\mathbf{x}}^{\text{L}} \xrightarrow{\sim} X(T/\mathbf{x}T)$$

and that it descends perfectly if it descends perfectly for all regular sequences  $\mathbf{x}$ . Note in particular that Nakayama's lemma implies that if  $X(\cdot)$  descends  $\mathbf{x}$ -perfectly and if  $X(T/\mathbf{x})$  is acyclic, then so is  $X(T)$ .

Let  $\mathbb{F}$  be the residue field of  $S$  and let  $I$  be an injective hull of  $\mathbb{F}$ . Let  $D$  be the Matlis duality functor  $D(\cdot) = \text{Hom}_S(\cdot, I)$ . If  $\mathbb{F}$  is finite, let  $D_p$  be the Pontrjagin duality functor  $D_p(\cdot) = \text{Hom}_{\mathbb{Z}_p}(\cdot, \mathbb{Q}_p/\mathbb{Z}_p)$ . If  $D$  and  $D_p$  are both defined, they coincide.

## 2.2.2 Greenberg Selmer groups

We assume that  $\mathbb{F}$  is finite. Let  $A$  be  $D_p(T) = T \otimes_S \text{Hom}_{\mathbb{Z}_p}(S, \mathbb{Q}_p/\mathbb{Z}_p)$ . For  $v|p$ , let  $A_v^+$  be  $D_p(T_v^+)$  and  $A_v^-$  be  $A/A_v^+$ . Let  $\text{Sel}(L, A)$  and  $\text{Sel}_{\text{str}}(L, A)$  be the co-finite  $S$ -modules defined respectively by the following exact sequences:

$$0 \longrightarrow \text{Sel}(L, A) \longrightarrow H^1(L, A) \longrightarrow \bigoplus_{v|p} H^1(I_v, A_v^-) \oplus \bigoplus_{v \in \Sigma \setminus \Sigma_p} H^1(I_v, A) \quad (2.2.1)$$

$$0 \longrightarrow \text{Sel}_{\text{str}}(L, A) \longrightarrow H^1(L, A) \longrightarrow \bigoplus_{v|p} H^1(G_{L_v}, A_v^-) \oplus \bigoplus_{v \in \Sigma \setminus \Sigma_p} H^1(I_v, A) \quad (2.2.2)$$

There is thus an exact sequence:

$$0 \longrightarrow \mathrm{Sel}_{\mathrm{str}}(L, A) \longrightarrow \mathrm{Sel}(L, A) \longrightarrow \bigoplus_{v|p} H^1(G_{L_v}/I_v, (A_v^-)^{I_v})$$

Or equivalently:

$$0 \longrightarrow \mathrm{Sel}_{\mathrm{str}}(L, A) \longrightarrow \mathrm{Sel}(L, A) \longrightarrow \bigoplus_{v|p} (A_v^-)^{I_v} / (\mathrm{Fr}(v) - 1)$$

Let  $\mathrm{Sel}(L, T)$  and  $\mathrm{Sel}_{\mathrm{str}}(L, T)$  be respectively  $D_p(\mathrm{Sel}(L, A))$  and  $D_p(\mathrm{Sel}_{\mathrm{str}}(L, A))$ .

**Lemma 2.7.** *Let  $\kappa$  be an arithmetic point such that  $\pi(f_\kappa)_v$  is not Steinberg at  $v|p$ . Then  $\mathrm{Sel}(F, T_\kappa)$  is isomorphic to  $\mathrm{Sel}_{\mathrm{str}}(F, T_\kappa)$*

*Proof.* □

### 2.2.3 Selmer complexes

Let  $A$  be  $D(T)$  and  $X$  be  $T$  or  $A$ . For  $v|p$ , let  $X_v^+$  and  $X_v^-$  be respectively  $T_v^+$  or  $D(T_v^+)$  and  $X/X_v^+$ . Let  $C_f^\bullet(L_v, T)$  be  $C_{\mathrm{cont}}^\bullet(L_v, X_v^+)$ . For  $v \nmid p$ , let  $C_f^\bullet$  be  $(\mathrm{Fr}(v), X^{I_v})$ . Let  $C_f^\bullet(L, X)$  be the complex of (co-)finite type  $S$ -modules:

$$C_f^\bullet(L, X) = \mathrm{Cone} \left( C_{\mathrm{cont}}^\bullet(L, X) \oplus \bigoplus_{v \in \Sigma} C_f^\bullet(L_v, X) \longrightarrow \bigoplus_{v \in \Sigma} C_{\mathrm{cont}}^\bullet(L_v, X) \right) [-1] \quad (2.2.3)$$

Let  $\mathrm{R}\Gamma_f(L, X)$  be the corresponding object in the derived category and let  $\tilde{H}_f^i(L, X)$  be the  $i$ -th cohomology group of  $\mathrm{R}\Gamma_f(L, X)$ . According to [Nek06, (8.9.6.1)], Matlis duality induces an isomorphism of complexes

$$\mathrm{R}\Gamma_f(L, T) \xrightarrow{\sim} D(\mathrm{R}\Gamma_f(L, A(1))) [-3]$$

inducing isomorphisms in cohomology:

$$\tilde{H}_f^i(L, T) \xrightarrow{\sim} D(\tilde{H}_f^{3-i}(L, A(1))) \quad (2.2.4)$$

If  $\mathbb{F}$  is finite, the following short sequence is exact:

$$0 \rightarrow \tilde{H}_f^0(L, A) \rightarrow H^0(L, A) \rightarrow \bigoplus_{v|p} H^0(L_v, A_v^-) \rightarrow \tilde{H}_f^1(L, A) \rightarrow \mathrm{Sel}_{\mathrm{str}}(L, A) \rightarrow 0 \quad (2.2.5)$$

If in addition  $H^0(L_v, A_v^-) = 0$  for all  $v|p$ , then (2.2.4) and (2.2.5) show that  $\tilde{H}_f^2(L, T)$  is isomorphic to  $\mathrm{Sel}_{\mathrm{str}}(L, T)$ .

**Proposition 2.8.** *The complex  $\mathrm{R}\Gamma_f(L, \mathcal{F})$  is quasi-isomorphic to the derived object corresponding to:*

$$C_f^\bullet(L, \mathcal{F}) = \mathrm{Cone} \left( C_{\mathrm{cont}}^\bullet(L, \mathcal{F}) \oplus \bigoplus_{v|p} C_f^\bullet(L_v, \mathcal{F}) \longrightarrow \bigoplus_{v \in \Sigma} C_{\mathrm{cont}}^\bullet(L_v, \mathcal{F}) \right) [-1] \quad (2.2.6)$$

*In particular, it descends perfectly.*

*Proof.* Let  $v$  be in  $\Sigma \setminus \{v|p\}$ . According to lemma 2.6, it is enough to show that  $C_f^\bullet(L_v, \mathcal{T})$  is acyclic. By inflation, it is thus enough to show that  $C_{\text{cont}}^\bullet(L_v, \mathcal{T})$  is acyclic. Let  $g$  be an arithmetic specialization of  $\mathcal{T}$  of parallel weight  $k \geq 4$ . Let  $\mathbf{x}$  be a regular sequence such that  $\mathcal{T}/\mathbf{x}$  is isomorphic to  $V(g)$ . According to lemma 2.6, it is enough to show that  $C_{\text{cont}}^\bullet(L_v, V(g))$  is acyclic. This last statement follows from purity of the  $G_{F_v}$ -representation  $V(f)$  as in [Nek06, Proposition 12.4.8.4].  $\square$

We note that the previous proof readily generalizes to  $S$ -specializations containing a regular specialization  $T$  such that  $C_{\text{cont}}^\bullet(L_v, T)$  is acyclic for all  $v \nmid p$ . In keeping with the analogies recorded in paragraph 2.1.3.5, we note the following proposition.

**Proposition 2.9.** *Let  $S$  be  $R_{\mathcal{P}}^{\text{min}}$  or  $R$  and  $T$  be  $\mathcal{T}_f$  or  $\mathcal{T}(\mathfrak{a})$ . The complex  $\text{R}\Gamma_f(L, T)$  is a perfect complex of  $S$ -modules concentrated in degrees 1 and 2.*

*Proof.* As  $T$  is a free  $S$ -module and as the  $p$ -cohomological dimension of  $G_{L, \Sigma}$  and  $G_{L_v}$  for all  $v$  are bounded by 2, the complex of  $S$ -modules  $\text{R}\Gamma_f(L, T)$  is perfect and concentrated in degrees  $[0, 3]$ . If  $T$  is equal to  $\mathcal{T}(\mathfrak{a})$ , absolute irreducibility of  $\mathcal{T}(\mathfrak{a})/\mathfrak{m}$  and (2.2.4) show that  $\tilde{H}_f^0(L, \mathcal{T}_{\mathfrak{m}})$  and  $\tilde{H}_f^3(L, \mathcal{T}_{\mathfrak{m}})$  are zero. If  $T$  is equal to  $\mathcal{T}_f$ , let  $\mathcal{D}$  denote Grothendieck's duality functor. As  $R_{\mathcal{P}}^{\text{min}}$  is a regular ring,  $\mathcal{D}(\cdot) = \text{Hom}_{R_{\mathcal{P}}^{\text{min}}}(\cdot, R_{\mathcal{P}}^{\text{min}})$ . Proposition 2.8 combined with [Nek06, (8.9.6.3)] show that:

$$\tilde{H}_f^3(L, \mathcal{T}_f) \xrightarrow{\sim} \mathcal{D}\left(\tilde{H}_f^0(L, \mathcal{D}(\mathcal{T}_f)(1))\right)$$

Hence  $\tilde{H}_f^0(L, \mathcal{T}_f)$  and  $\tilde{H}_f^3(L, \mathcal{T}_f)$  are zero.  $\square$

### 3 Control theorems

#### 3.1 The general case

**Proposition 3.1.** *Let  $V(f)$  be an arithmetic specialization of  $\mathcal{V}$ . Then the rank of  $\tilde{H}_f^j(L, \mathcal{V})$  is bounded by the rank of  $\tilde{H}_f^j(L, V(f))$  for  $j = 1, 2$ .*

*Proof.* Let  $\mathcal{P}$  be the arithmetic prime corresponding to  $V(f)$ . Then  $\mathcal{V}$  contains an  $R_{\mathcal{P}}^{\text{min}}$ -representation  $\mathcal{T}_f$  such that  $\mathcal{T}_f/\mathcal{P}\mathcal{T}_f$  is isomorphic to  $V(f)$ . As  $R_{\mathcal{P}}^{\text{min}}$  is a regular ring, there exists a regular sequence  $\mathbf{x} = (x_1, \dots, x_r)$  generating  $\mathcal{P}$ . For  $i \geq 1$ , let  $\mathbf{x}_i$  be  $(x_1, \dots, x_i)$ , let  $R_i$  be  $R_{\mathcal{P}}^{\text{min}}/\mathbf{x}_i$  and let  $T_i$  be  $\mathcal{T}_f/\mathbf{x}_i$ . According to proposition 2.8, there is an isomorphism of complexes

$$\text{R}\Gamma_f(L, T_i) \otimes^L R_i/x_{i+1}R_i \xrightarrow{\sim} \text{R}\Gamma_f(L, T_{i+1})$$

and thus an injection of cohomology groups (which is actually an isomorphism for  $j = 2$  by proposition 2.9):

$$\tilde{H}_f^j(L, T_j)/x_{i+1}\tilde{H}_f^j(L, T_i) \hookrightarrow \tilde{H}_f^j(L, T_{i+1})$$

So the  $R_i$ -rank of  $\tilde{H}_f^j(L, T_i)$  bounds below the  $R_{i+1}$ -rank of  $T_{i+1}$ .  $\square$

Again, we note that this proof generalizes to  $S$ -specializations containing a specialization  $T$  such that  $C_{\text{cont}}^\bullet(L_v, T)$  is acyclic for all  $v \nmid p$ .

### 3.2 The regular absolutely irreducible case

In this subsection, we fix a nearly ordinary eigenform  $f$  such that  $\bar{\rho}_f$  is irreducible and use the constructions of paragraph 2.1.3.3. In addition, we assume that the ring  $R$  is regular. For any given nearly ordinary eigenform  $f$ , we observe that these conditions are satisfied for sufficiently large  $p$  by [Dim05, Proposition 0.1] and lemma 2.3.

**Proposition 3.2.** *Let  $\mathcal{P} \in \text{Spec}^{\text{arith}}(R)$  be an arithmetic prime of  $R$  and  $T(f)$  be the corresponding Galois representation. Then:*

$$\mathrm{R}\Gamma_f(L, \mathcal{T}) \otimes^{\mathrm{L}} R/\mathcal{P} \xrightarrow{\sim} \mathrm{R}\Gamma_f(L, T(f)) \quad (3.2.1)$$

Moreover:

$$\tilde{H}_f^2(L, \mathcal{T})/\mathcal{P}\tilde{H}_f^2(L, \mathcal{T}) \xrightarrow{\sim} \tilde{H}_f^2(L, T(f)) \quad (3.2.2)$$

*Proof.* As  $R$  is a regular ring, there exists an  $R$ -regular sequence  $\mathbf{x}$  generating  $\mathcal{P}$ . According to lemma 2.6, in order to prove (3.2.1), it is enough to prove that  $\mathrm{R}\Gamma_f(L, \mathcal{T})$  descends  $\mathbf{x}$ -perfectly. Hence it is enough to prove that  $\mathrm{R}\Gamma_f(L_v, \mathcal{T})$  descends  $\mathbf{x}$ -perfectly for all  $v$  dividing  $\mathcal{N}$ . Fix such a  $v$ . As in the proof of proposition 3.1, let  $\mathbf{x}_i$  be the regular sub-sequence  $(x_1, \dots, x_i)$  and let  $T_i$  be  $\mathcal{T}/\mathbf{x}_i$ . By lemma 2.5,  $T(f)^{I_v}$ ,  $T_i^{I_v}$  and  $T_{i+1}^{I_v}$  are all free of rank 1. Thus,  $T_i^{I_v}$  surjects onto  $T_{i+1}^{I_v}$  for all  $i \geq 0$ . The short exact sequence

$$0 \longrightarrow T_i^{I_v} \xrightarrow{x_{i+1}} T_i^{I_v} \longrightarrow T_{i+1}^{I_v} \longrightarrow 0$$

shows that there is an isomorphism of complexes  $\mathrm{R}\Gamma_f(L, T_i) \otimes^{\mathrm{L}} R_i/x_{i+1} \xrightarrow{\sim} \mathrm{R}\Gamma_f(L, T_{i+1})$ . This concludes the proof of (3.2.1).

The statement (3.2.2) follows from proposition 2.9 and the exact sequence in cohomology corresponding to (3.2.1).  $\square$

**Corollary 3.3.** *Under the hypotheses of 3.2, there is an isomorphism*

$$\mathrm{Sel}_{\mathrm{str}}(L, \mathcal{T}(\mathbf{a})) \otimes_R R/\mathcal{P}_f \xrightarrow{\sim} \mathrm{Sel}_{\mathrm{str}}(L, T(f)) \quad (3.2.3)$$

*at all arithmetic specializations  $T(f)$  such that the automorphic representation  $\pi(f)_v$  is not Steinberg at  $v|p$ . Under the same hypothesis on  $\pi(f)$ , the map*

$$\mathrm{Sel}(L, \mathcal{T}(\mathbf{a})) \otimes_R R/\mathcal{P}_f \longrightarrow \mathrm{Sel}(L, T(f)) \quad (3.2.4)$$

*has finite kernel and cokernel.*

*Proof.* Under the hypotheses of 3.2, proposition 3.2 and the isomorphism between  $\mathrm{Sel}_{\mathrm{str}}(L, \mathcal{T})$  and  $\tilde{H}_f^2(L, \mathcal{T})$  establish (3.2.3).  $\square$

## 4 Applications

### 4.1 Examples of torsion Selmer groups

Let  $\mathcal{T}_f$  be as in paragraph 2.1.3.2 and let  $\mathcal{V}$  be the associated  $\mathrm{Frac}(R^{\min})$ -vector space. If there exists a specialization  $V$  of  $\mathcal{V}$  such that  $\tilde{H}_f^2(L, V)$  is zero and such that  $C_{\mathrm{cont}}^{\bullet}(L_v, V)$  is acyclic for all  $v \nmid p$ , proposition 3.1 implies that  $\tilde{H}_f^2(L, \mathcal{T}_f)$  is torsion. The hypotheses are in particular verified if  $\mathcal{V}$  admits an arithmetic specialization of parallel weight  $V(f)(k/2)$  such that  $\tilde{H}_f^2(L, V(f)(k/2))$  is trivial. Likewise, under the hypotheses of subsection 3.2, if  $\mathcal{T}(\mathbf{a})$  admits an arithmetic specialization  $T(f)(k/2)$  such that  $\tilde{H}_f^2(L, T(f)(k/2))$  is trivial (resp. torsion), then  $\tilde{H}_f^2(L, \mathcal{T}(\mathbf{a}))$  is trivial (resp. torsion).

**Proposition 4.1.** *Assume that there exists an arithmetic specialization  $V(f)$  of  $\mathcal{V}$  (resp. of  $\mathcal{T}(\mathfrak{a})$ ) if  $\bar{\rho}_{\mathfrak{m}}$  is irreducible) such that:*

1. *The representation  $\pi(f)$  is of parallel weight  $(2, \dots, 2)$  and not Steinberg.*
2. *There exists a character  $\chi$  of  $\mathbb{A}_F^\times/F^\times$  such that  $\chi^2 = \omega(f)^{-1}$ .*
3. *The sign  $\epsilon(\pi(f), 1/2)$  of the functional equation of  $L(V(f)(1) \otimes \chi, s)$  is equal to 1.*

*Then  $\tilde{H}_f^2(F, \mathcal{T}_f)$  (resp.  $\tilde{H}_f^2(F, \mathcal{T}(\mathfrak{a}))$ ) is torsion.*

*Proof.* Let  $T$  be  $\mathcal{T}_f$  or  $\mathcal{T}(\mathfrak{a})$ . Let  $V(f)^\dagger$  be the self-dual  $G_F$ -representations  $V(f)(1) \otimes \chi$ . According to [FH95, Theorem B], there exists a quadratic totally complex extension  $K/F$  such that the Rankin-Selberg  $L$ -function  $L(V(f)^\dagger \otimes \eta_{K/F}, s)$  vanishes at exact order 1 at  $s = 1$ . According to [YZZ08, Theorem 1.4.1] due to Y.Tian and S.Zhang, this implies that  $\tilde{H}_f^1(K_\Sigma/K, V(f)(1))$  is of dimension 1. This and hypothesis 3 of the proposition show by [Nek06, Corollary 12.2.3] that the dimension of  $\tilde{H}_f^1(F, V(f)^\dagger)$  is even. The vector space  $\tilde{H}_f^1(F, V(f)^\dagger)$  is then of even dimension smaller than 1, and thus trivial. This establishes the proposition if  $T$  is equal to  $\mathcal{T}_f$ . Localizing at  $\mathcal{P}_f$  then shows the result if  $T$  is equal to  $\mathcal{T}(\mathfrak{a})$  □

**Corollary 4.2.** *Under the hypotheses of proposition 4.1 and if  $\bar{\rho}_{\mathfrak{m}}$  is irreducible, the  $R$ -modules  $\text{Sel}_{\text{str}}(L, \mathcal{T}(\mathfrak{a}))$  and  $\text{Sel}(F, \mathcal{T}(\mathfrak{a}))$  are torsion.*

**Corollary 4.3.** *Let  $T$  be  $\mathcal{T}_f$  or  $\mathcal{T}(\mathfrak{a})$ . Under the hypotheses of proposition 4.1,  $\text{R}\Gamma_f(L, T)$  is acyclic outside 2.*

*Proof.* According to propositions 3.1 and 4.1,  $\tilde{H}_f^1(L, T)$  is a torsion module while according to proposition 2.9, it is a submodule of a free module. Hence it vanishes. □

## 4.2 Iwasawa main conjecture

We assume the hypotheses of subsection 3.2 as well as the hypotheses of proposition 4.1. Then the complex  $\text{R}\Gamma_f(F, \mathcal{T}(\mathfrak{a}))$  is a perfect complex of torsion  $R$ -modules concentrated in degree 2. The acyclicity of  $\text{R}\Gamma_f(F, \mathcal{T}(\mathfrak{a})) \otimes_R \text{Frac}(R)$  defines a canonical isomorphism:

$$\zeta_R : \text{Frac}(R) \xrightarrow{\sim} \det_R^{-1} \text{R}\Gamma_f(F, \mathcal{T}(\mathfrak{a})) \otimes_R \text{Frac}(R)$$

Because the ring  $R$  is regular, the image of  $\det_R^{-1} \text{R}\Gamma_f(F, \mathcal{T}(\mathfrak{a}))$  inside  $\text{Frac}(R)$  under  $\zeta_R^{-1}$  is the ideal of  $R$  generated by  $\text{char}_R \tilde{H}_f^2(F, \mathcal{T}(\mathfrak{a}))$ .

The philosophy recalled in the introduction predicts that there exists a  $p$ -adic  $L$ -function  $L_p(\mathcal{T}(\mathfrak{a}))$  which is a non-zero element of  $R$  interpolating the critical values of  $L(V(f), 0)$  divided by a suitable period and modified by Euler factors at  $v|p$ . Under our working hypotheses, this  $p$ -adic  $L$ -function is constructed in forthcoming work of M.Dimitrov and one of us. The Iwasawa Main Conjecture in this setting predicts that the image of  $L_p(\mathcal{T}(\mathfrak{a}))$  under  $\zeta_R$  is a basis of the  $R$ -lattice  $\det_R^{-1} \text{R}\Gamma_f(F, \mathcal{T}(\mathfrak{a}))$  inside  $\text{Frac}(R)$ , or equivalently that:

$$\text{char}_R \tilde{H}_f^2(F, \mathcal{T}(\mathfrak{a})) = (L_p(\mathcal{T}(\mathfrak{a}))) \tag{4.2.1}$$

For  $\kappa$  an arithmetic point, let  $C(\kappa)L(V_\kappa, 0)$  be the value of  $\kappa(L_p(\mathcal{T}(\mathfrak{a})))$ . Let  $\kappa$  be an arithmetic point with values in  $S$  such that  $\kappa(L_p(\mathcal{T}(\mathfrak{a})))$  is non-zero and such that  $\text{R}\Gamma_f(F, T_\kappa) \otimes_S \text{Frac}(S)$  is acyclic. We note that the set of such  $\kappa_f$  is infinite under our hypotheses. The Tamagawa Number Conjecture predicts that  $\zeta_R \otimes_{R, \kappa} S$  induces an isomorphism

$$\zeta_{S, \kappa} : \text{Frac}(S) \xrightarrow{\sim} \det_S^{-1} \text{R}\Gamma_f(F, T_\kappa) \otimes \text{Frac}(S)$$

such that the image of  $\kappa(L_p(\mathcal{T}(\mathfrak{a})))$  is a basis of the  $S$ -lattice  $\det_S^{-1} \mathrm{R}\Gamma_f(F, T_\kappa)$  inside  $\mathrm{Frac}(S)$  identified with  $\det_S^{-1} \mathrm{R}\Gamma_f(F, T_\kappa) \otimes_S \mathrm{Frac}(S)$ .

**Proposition 4.4.** *Assume that the Iwasawa Main Conjecture (4.2.1) is true for  $\mathcal{T}(\mathfrak{a})$ . Let  $\kappa$  be an arithmetic specialization such that  $C(\kappa)L(V_\kappa, 0)$  is non-zero and such that the complex  $\mathrm{R}\Gamma_f(F, T_\kappa) \otimes_S \mathrm{Frac}(S)$  is acyclic. Then the Tamagawa Number Conjecture for  $T_\kappa$  is true.*

*Proof.* As the image of  $\kappa(L_p(\mathcal{T}(\mathfrak{a})))$  by  $\zeta_{S, \kappa}$  is equal to  $\zeta_R(\mathcal{T}(\mathfrak{a})) \bmod \ker \kappa$ , the statement of the proposition is implied by the isomorphism (3.2.1) of proposition 3.2:

$$\mathrm{R}\Gamma_f(F, \mathcal{T}(\mathfrak{a})) \otimes_R^L R / \ker \kappa \xrightarrow{\sim} \mathrm{R}\Gamma_f(F, T_\kappa)$$

□

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