# DIAGONAL SUPERCOMPACT RADIN FORCING

OMER BEN-NERIA, CHRIS LAMBIE-HANSON AND SPENCER UNGER

## 1. Introduction

In this paper we consider two methods for producing models with some global behavior of the continuum function on singular cardinals and the failure of weak square. The first method is as an extension of Sinapova's work [21]. We define a diagonal supercompact Radin forcing which adds a club subset to a cardinal $\kappa$ while forcing the failure of SCH everywhere on the club. The intuition from Sinapova's work was that our model should have the failure of weak square at every point in the club. Unfortunately, the additional closure required to preserve inaccessibility enforces weak square at some points.

**Theorem 1.1.** *If there are a supercompact cardinal $\kappa$ and a weakly inaccessible $\theta > \kappa$, then there is a forcing extension in which $\kappa$ is regular and there are a club $C \subseteq \kappa$ of singular cardinals where SCH fails, and $\square^*_\nu$ fails for every $\nu \in C \setminus \mathrm{cof}(\omega)$.*

In the second method, we use Gitik's method of iterating Prikry type posets to iterate Sinapova's poset and obtain the following theorem.

**Theorem 1.2.** *Let $\kappa$ be a regular cardinal. If for all regular $\tau < \kappa$ the set $\{\nu < \kappa \mid \nu$ is $\nu^{+\tau}$-supercompact $\}$ is stationary, then there is a forcing extension in which $\kappa$ is inaccessible and there is a club $C$ in $\kappa$ such that for all $\kappa \in C$ both $\square^*_\kappa$ and SCH fail.*

Although the second theorem is formally better than the first, we believe that the techniques from the proof are unlikely to generalize.

We are motivated by the question of whether in ZFC one can construct a $\kappa$-Aronszajn tree for some $\kappa > \omega_1$. The question is also open if we ask for a *special* $\kappa$-Aronszajn tree. Forcing provides a possible path to a negative solution by showing that it is consistent with ZFC that there are no $\kappa$-Aronsajn trees on any $\kappa > \omega_1$. By a theorem of Jensen [**?**], $\square^*_\mu$ is equivalent to the existence of a special $\mu^+$-Aronszajn tree. So our theorems are partial progress towards a model with no special Aronszajn trees.

The non-existence of $\kappa$-Aronszajn trees (the tree property at $\kappa$) and the non-existence of special $\kappa$-Aronszajn trees (failure of $\square^*$) are reflection principles which are closely connected with large cardinals. For example, theorems of Erdös and Tarski [6], and Monk and Scott [17], show that an inaccessible cardinal is weakly compact if and only if it has the tree property. Further Mitchell and Silver [16] showed that the tree property at $\aleph_2$ is consistent with ZFC if and only if the existence of a weakly compact cardinal is.

Specker [25] showed that if $\kappa^{<\kappa} = \kappa$, then there is a special $\kappa^+$-Aronszajn tree. This theorem places an important restriction on models where there are no special

*Date*: February 9, 2016.

Aronszajn trees. From Specker's theorem, a model with no special $\kappa$-Aronszajn trees for any $\kappa > \aleph_1$ must be one in which GCH fails everywhere. In particular GCH must fail at every singular strong limit cardinal, a failure of the Singular Cardinals Hypothesis (SCH). The consistency of the failure of SCH requires large cardinals [10] and a model in which GCH fails everywhere was first obtained by Foreman and Woodin [8].

There are many partial results towards constructing a model in which every regular cardinal greater than $\aleph_1$ has the tree property. There is a bottom up approach where one attempts to force longer and longer initial segments of the regular cardinals to have the tree property, see for example ([1, 3, 19, 26]). We refer the reader to [?] for some analogous results on successive failures of weak square. Another aspect of the problem comes from the interaction between cardinal arithmetic at singular strong limit cardinals $\mu$ and the tree property at $\mu^+$. In the 1980's Woodin asked whether the failure of SCH at $\aleph_\omega$ is consistent with the tree property at $\aleph_{\omega+1}$. More generally one can consider whether this situation is consistent at some larger singular cardinal. An important result in this direction is due to Gitik and Sharon [12] who showed that from the existence of a supercompact cardinal that it is consistent that there is a singular cardinal $\kappa$ of cofinality $\omega$ such that SCH fails at $\kappa$ and there are no special $\kappa^+$-Aronszajn trees. In fact they show a stronger assertion ($\kappa^+ \notin I[\kappa^+]$), which we will define later. In the same paper, they show that it is possible to make $\kappa$ in to $\aleph_{\omega^2}$. Cummings and Foreman [4] showed that there is a PCF theoretic object called a bad scale in the models of Gitik and Sharon which implies that $\kappa^+ \notin I[\kappa^+]$.

The key ingredient Gitik and Sharon's argument was a new diagonal supercompact Prikry forcing. The basic idea is to start with supercompactness measures $U_n$ on $\mathcal{P}_\kappa(\kappa^{+n})$ for $n < \omega$ and use it to define a Prikry forcing. This forcing adds a sequence $\langle x_n \mid n < \omega\rangle$ where each $x_n$ is a typical point for $U_n$ and so that $\bigcup_{n<\omega} x_n = \kappa^{+\omega}$. The result is that $\kappa^{+\omega}$ is collapsed to have to have size $\kappa$ and $\kappa^{+\omega+1}$ becomes the new successor of $\kappa$. This allows for the property $\kappa^{+\omega+1} \notin I[\kappa^{+\omega+1}]$ from the ground model to show $\kappa^+ \notin I[\kappa^+]$ in the extension. Moreover if we started with $2^\kappa = \kappa^{+\omega+2}$ in the ground model, then we get the failure of SCH at $\kappa$ in the extension.

Variations of Gitik and Sharon's poset have been used construct many related models. We list a few such results:

(1) (Neeman [18]) From $\omega$ supercompact cardinals, there is a forcing extension in which there is a singular cardinal $\kappa$ of cofinality $\omega$ such that SCH fails at $\kappa$ and $\kappa^+$ has the tree property.
(2) (Sinapova [21]) From a supercompact cardinal $\kappa$, for any regular $\lambda < \kappa$, there is a forcing extension in which $\kappa$ is a singular cardinal of cofinality $\lambda$, SCH fails at $\kappa$ and $\kappa$ carries a bad scale (in particular $\kappa^+ \notin I[\kappa^+]$ and there are no special $\kappa^+$-Aronszajn trees).
(3) (Sinapova [22]) From $\lambda$ supercompact cardinals $\langle \kappa_\alpha \mid \alpha < \lambda\rangle$ with $\lambda < \kappa_0$ regular, there is a forcing extension in which $\kappa_0$ is a singular cardinal of cofinality $\lambda$, SCH fails at $\kappa_0$ and $\kappa_0^+$ has the tree property.
(4) (Sinapova [23]) From $\omega$ supercompact cardinals it is consistent that Neeman's result above holds with $\kappa = \aleph_{\omega^2}$.

Woodin's original question remains open, see [24] for the best known partial result. A theme in the above results is that questions about the tree property are answered by first constructing a model where there are no special $\kappa^+$-Aronszajn trees (or even $\kappa^+ \notin I[\kappa^+]$). To obtain the tree property, one needs to increase the large cardinal assumption and to give a version of an argument of Magidor and Shelah [15] who showed that the tree property holds at $\mu^+$ when $\mu$ is a singular limit of supercompact cardinals.

In this paper we pursue a global version of the results above. We aim to force a class club $C$ of singular cardinals such that for all $\kappa \in C$ SCH fails at $\kappa$ and there are no special $\kappa^+$-Aronszajn trees. From Specker's theorem, a model with no Aronszajn trees above $\aleph_1$ (or even no special Aronszajn trees above $\aleph_1$) will have such a club $C$. Our attempt to construct the natural diagonal supercompact Radin forcing does not give the desired class $C$. We succeed in forcing a club $C$ of singular cardinals where SCH fails, but some successors of elements of $C$ carry special Aronszajn trees. In the last section of the paper, we sketch the construction of a model where we do get the class club $C$ that we wanted, but the technique used looks unlikely to generalize to help answer our motivating question.

To put the theorems of the paper in the context of the results above we need a brief description of Magidor forcing and Radin forcing. Changing the cofinality of a large cardinal to some prescribed uncountable regular cardinal was first done by Magidor [14] and the technique has come to be known as Magidor forcing. Sinapova's results about uncountable cofinalities mentioned above are done by creating a Magidor forcing version of the Gitik-Sharon poset. Adding a Prikry-like club in a large cardinal $\kappa$ while preserving its regularity was first done by Radin [20]. In this paper we describe a diagonal supercompact Radin forcing which adds a Prikry-like club of cardinals where SCH fails. Unfortunately, the extra closure we need to preserve the inaccessibility of our large cardinal $\kappa$ interfers with the argument for the non-existence of special Aronszajn trees at (the successors of) some elements of the club. There are strong reasons to believe that Radin forcing is the correct technique for solving our motivating question. In particular, Radin forcing can admit the addition for forcing between the elements of the generic club, see for example [13, 8, 2, 7].

In the final section of the paper we use Gitik's idea (See [11] for a reference) of iterating Prikry type forcing to show that it is consistent to have a class club $C$ as above. However this technique does not seem to help us with our motivating question and the ideas are mostly standard, so we only sketch the proof.

The paper is organized as follows. In Section 2 we give some definitions and background material required for the main result. In section 3 we describe the main forcing and prove that it gives a model with a class club $C$ of cardinals where SCH fails. In Section 4 we show that there is a stationary, co-stationary set $S \subseteq C$ such that for all $\kappa \in S$ $\kappa^+ \notin I[\kappa^+]$ and for all $\kappa \in C \setminus S$ there is a special $\kappa^+$-Aronszajn tree. In Section 5 we show obtain a class club of cardinals $\kappa$ where SCH fails and there are no special $\kappa^+$-Aronszajn trees. In Section **??** we make some concluding remarks and ask some open questions.

## 2. Background

In this section we will make the notions from the introduction precise and give some further definitions which are relevant to the rest of the paper.

**Definition** We define a weak square sequence $\square^*_\nu$ and the approachability ideal $I[\nu^+]$ for a cardinal $\nu$.

- We say that $\nu$ *has a weak square sequence* ($\square^*_\nu$) $\langle C_\gamma \mid \gamma < \nu^+\rangle$ is a partial square sequence if
  1. for all $\gamma < \nu^+$ limit, $C_\gamma \subset \mathcal{P}(\gamma)$ has size $|C_\gamma| \leq \nu$ such that for every $c \in C_\gamma$, $c \subset \gamma$ is club in $\gamma$, $\mathrm{otp}(C_\gamma) < \nu$, and
  2. for all $\beta < \gamma < \nu$ if $\beta$ is a limit point of some $c \in C_\gamma$ then $c \cap \beta \in C_\beta$.
- Let $\vec{z} = \langle z_\alpha \mid \alpha < \nu^+\rangle$ be a sequence of bounded subsets of $\nu^+$. We say that a limit ordinal $\gamma$ is $\vec{z}-$approachable if there is a closed unbounded set $A \subset \gamma$ with $otp(A) = cf(\gamma)$ such that for every $\beta < \gamma$, $A \cap \beta \in \vec{z} \restriction \gamma = \langle z_\alpha \mid \alpha < \gamma\rangle$. By arranging that $z_{\alpha+1}$ is the closure of $z_\alpha$ for each $\alpha < \nu^+$ we may assume that for every $\vec{z}-$approachable points $\gamma$, there is a witness $A \subset \gamma$ which is closed unbounded.
  The approachability ideal $I[\nu^+]$ consists of all subsets $S \subset \nu^+$ for which there are $\vec{z}$ as above and a club $C \subset \nu^+$ so that every $\gamma \in C \cap S$ is $\vec{z}-$approachable.

2.1. **Forcing preliminaries.** Suppose $\kappa$ is supercompact and $\theta > \kappa$. As in [21], we can arrange that $2^\kappa \geq \theta$ and that we have a sequence of ultrafilters $\vec{U} = \langle U_\alpha \mid \alpha < \theta\rangle$ and, for all $\alpha < \theta$, a sequence $\langle f^\alpha_\eta \mid \eta < \theta\rangle$ such that the following hold.

- For all $\alpha < \theta$, $U_\alpha$ is a normal, fine ultrafilter on $\mathcal{P}_\kappa(\kappa^{+\alpha})$. Let $j_\alpha : V \to M_\alpha \cong \mathrm{Ult}(V, U_\alpha)$ be the collapsed ultrapower map.
- For all $\alpha < \beta < \theta$, $U_\alpha \in M_\beta$.
- For all $\alpha, \eta < \theta$, we have $f^\alpha_\eta : \kappa \to \kappa$ and $j_\alpha(f^\alpha_\eta)(\kappa) = \eta$.

When we write that something happens for most $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, we mean it happens for a $U_\alpha$-measure one set. For $\alpha < \theta$ and $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, let $\kappa_x$ denote $x \cap \kappa$. For most $x \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, $x \cap \kappa$ is an inaccessible cardinal. We will always assume we are working with such $x$ and let $\kappa_x = x \cap \kappa$. For $x, y \in \mathcal{P}_\kappa(\kappa^{+\alpha})$, $x \prec y$ denotes the statement that $x \subseteq y$ and $\mathrm{otp}(x) < \kappa_y$.

For $\alpha < \beta < \theta$, let $\bar{u}^\beta_\alpha$ be a function on $\mathcal{P}_\kappa(\kappa^{+\beta})$ representing $U_\alpha$ in the ultrapower by $U_\beta$. For most $x \in \mathcal{P}_\kappa(\kappa^{+\beta})$, $\bar{u}^\beta_\alpha(x)$ is a measure on $\mathcal{P}_{\kappa_x}(\kappa_x^{+f^\beta_\alpha(\kappa_x)})$. Also, for most $x \in \mathcal{P}_\kappa(\kappa^{+\beta})$, $\mathrm{otp}(x \cap \kappa^{+\alpha}) = f^\beta_\alpha(\kappa_x)$. For such $x$, $\bar{u}^\beta_\alpha(x)$ lifts naturally to a measure $u^\beta_\alpha(x)$ on $\mathcal{P}_{\kappa_x}(x \cap \kappa^{+\alpha})$.

For $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, let $Z^\beta_y = \{\alpha < \beta \mid \kappa^{+\alpha} \in y\}$.

**Lemma 2.1.** *For most $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, the following hold.*

1. *$Z^\beta_y$ is $< \kappa_y$-closed.*
2. *If $\mathrm{cf}(\beta) < \kappa$, then $Z^\beta_y$ is unbounded in $\beta$.*
3. *$\mathrm{otp}(Z^\beta_y) = f^\beta_\beta(\kappa_y)$ and, if $\beta$ is a limit ordinal, then so is $f^\beta_\beta(\kappa_y)$. Also, if $cf(\beta) \geq \kappa$ then $\mathrm{cf}(f^\beta_\beta(\kappa_y)) \geq \kappa_y$.*
4. *For all $\alpha \in Z^\beta_y$, $\mathrm{otp}(y \cap \kappa^{+\alpha}) = f^\beta_\alpha(\kappa_y) = \kappa_y^{+\mathrm{otp}(\alpha \cap Z^\beta_y)}$.*
5. *For all $\alpha \in Z^\beta_y$, $\bar{u}^\beta_\alpha(y)$ is a measure on $\mathcal{P}_{\kappa_y}(\kappa_y^{+f^\beta_\alpha(\kappa_y)})$.*
6. *For all $\alpha_0 < \alpha_1$, both in $Z^\beta_y$, the function $x \mapsto \bar{u}^{\alpha_1}_{\alpha_0}(x)$ represents $\bar{u}^\beta_{\alpha_0}(y)$ in the ultrapower by $u^\beta_{\alpha_1}(y)$.*

*Proof.* Let $j = j_\beta$. Note first that, defining $g : \mathcal{P}_\kappa(\kappa^{+\beta}) \to V$ by $g(y) = Z^\beta_y$, we have $j(g)(j``\kappa^{+\beta}) = j``\beta$. Items (1)-(4) then follow easily.

To show (5), let $j(\langle \bar{u}^\beta_\alpha \mid \alpha < \beta\rangle) = \langle \bar{v}^{j(\beta)}_\alpha \mid \alpha < j(\beta)\rangle$ and $j(\langle f^\beta_\alpha \mid \alpha < \beta\rangle) = \langle g^{j(\beta)}_\alpha \mid \alpha < j(\beta)\rangle$. Let $\alpha \in j``\beta$, with, say, $\alpha = j(\xi)$. Then $\bar{v}^{j(\beta)}_\alpha(j``\kappa^{+\beta}) =$

$j(\bar{u}^\beta_\xi)(j``\kappa^{+\beta}) = U_\xi$, which is a measure on $\mathcal{P}_\kappa(\kappa^{+\xi}) = \mathcal{P}_\kappa(\kappa^{+g^{j(\beta)}_\alpha(\kappa)})$. Thus, for almost all $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$, $\bar{u}^\beta_\alpha(y)$ is a measure on $\mathcal{P}_{\kappa_y}(\kappa_y^{+f^\beta_\alpha(\kappa_y)})$.

We finally show (6). Let $j(\langle \bar{u}^{\alpha_1}_{\alpha_0} \mid \alpha_0 < \alpha_1 \leq \beta \rangle) = \langle \bar{v}^{\alpha_1}_{\alpha_0} \mid \alpha_0 < \alpha_1 \leq j(\beta)\rangle$ and $j(\langle u^\beta_\alpha \mid \alpha < \beta \rangle) = \langle v^{j(\beta)}_\alpha \mid \alpha < j(\beta)\rangle$. It suffices to show that, in $M_\beta$, for all $\alpha_0 < \alpha_1$, both in $j``\beta$, the function $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents $\bar{v}^{j(\beta)}_{\alpha_0}(j``\kappa^{+\beta})$ in the ultrapower by $v^{j(\beta)}_{\alpha_1}(j``\kappa^{+\beta})$. Note that $v^{j(\beta)}_{\alpha_1}(j``\kappa^{+\beta})$ is a measure on $\mathcal{P}_\kappa(j``\kappa^{+\xi_1})$ that collapses to $U_{\xi_1}$. Call this 'lifted' measure $\hat{U}_{\xi_1}$. Also note that $\bar{v}^{j(\beta)}_{\alpha_0}(j``\kappa^{+\beta}) = U_{\xi_0}$. Thus, we must show that the function $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents $U_{\xi_0}$ in the ultrapower by $\hat{U}_{\xi_1}$.

Fix $\alpha_0 < \alpha_1$ in $j``\beta$, with $\alpha_0 = j(\xi_0)$ and $\alpha_1 = j(\xi_1)$. Fix $x \in \mathcal{P}_\kappa(j``\kappa^{+\xi_1})$. There is $\bar{x} \in \mathcal{P}_\kappa(\kappa^{+\beta})$ such that $x = j(\bar{x})$. Then $\bar{v}^{\alpha_1}_{\alpha_0}(x) = j(\bar{u}^{\xi_1}_{\xi_0}(\bar{x}))$. For most $\bar{x} \in \mathcal{P}_\kappa(\kappa^{+\xi_1})$, $\bar{u}^{\xi_1}_{\xi_0}(\bar{x})$ is a measure on $\mathcal{P}_{\kappa_{\bar{x}}}(\kappa_{\bar{x}}^{+f^{\xi_1}_{\xi_0}(\kappa_{\bar{x}})})$, and this is fixed by $j$. Thus, for most $x \in \mathcal{P}_\kappa(j``\kappa^{+\xi_1})$, $\bar{v}^{\alpha_1}_{\alpha_0}(x) = \bar{u}^{\xi_1}_{\xi_0}(\bar{x})$. Thus, since $\hat{U}_{\xi_1}$ is a lifting of $U_{\xi_1}$, $x \mapsto \bar{v}^{\alpha_1}_{\alpha_0}(x)$ represents the same thing in the ultrapower by $\hat{U}_{\xi_1}$ as $\bar{x} \mapsto \bar{u}^{\xi_1}_{\xi_0}(x)$ represents in the ultrapower by $U_{\xi_1}$, which is $U_{\xi_0}$. This is true in $V$ and, since $M_\beta$ is sufficiently closed, it is true in $M_\beta$ as well. □

**Lemma 2.2.** *Suppose $\beta < \theta$ and, for all $\alpha \leq \beta$, $A_\alpha \in U_\alpha$. Let $A^*$ be the set of all $y \in A_\beta$ such that, for all $\alpha \in Z^\beta_y$, $\{x \in A_\alpha \mid x \prec y\} \in u^\beta_\alpha(y)$. Then $A^* \in U_\beta$.*

*Proof.* Let $j = j_\beta$. It suffices to show that $j``\kappa^{+\beta} \in j(A^*)$, i.e. for all $j(\alpha) \in j``\beta$, $\{x \in j(A_\alpha) \mid x \prec j``\kappa^{+\beta}\} \in \hat{U}_\alpha$, where $\hat{U}_\alpha$ is the lifted version of $U_\alpha$ living on $\mathcal{P}_\kappa(j``\kappa^{+\alpha})$. Fix such a $j(\alpha)$. Let $X = \{x \in j(A_\alpha) \mid x \prec j``\kappa^{+\beta}\}$, and note that $X = j``A \in \hat{U}_\alpha$. □

**Lemma 2.3.** *Suppose $\gamma < \theta$, $z \in \mathcal{P}_\kappa(\kappa^{+\gamma})$, and $z$ satisfies all of the statements in Lemma 2.1. Suppose that, for all $\alpha \in Z^\gamma_z$, $A_\alpha \in u^\gamma_\alpha(z)$. Fix $\beta \in Z^\gamma_z$, and let $A$ be the set of $y \in A_\beta$ such that, for all $\alpha \in Z^\beta_y$, $\{x \in A_\alpha \mid x \prec y\} \in u^\beta_\alpha(y)$. Then $A^* \in u^\gamma_\beta(z)$.*

*Proof.* For each $\alpha \in Z^\gamma_z$, let $\bar{A}_\alpha$ be the collapsed version of $A_\alpha$, so $\bar{A}_\alpha \in \bar{u}^\gamma_\alpha(z)$. Recall that $u^\gamma_\beta(z)$ is a measure on $\mathcal{P}_{\kappa_z}(z \cap \kappa^{+\beta})$. Let $k : V \to N \cong \mathrm{Ult}(V, u^\gamma_\beta(z))$ be the ultrapower map. By (6) of Lemma 2.1, for all $\alpha \in Z^\gamma_z \cap \beta$, the map $y \mapsto \bar{u}^\beta_\alpha(y)$ represents $\bar{u}^\gamma_\alpha(z)$ in the ultrapower. Note also that the map $y \mapsto Z^\beta_y$ represents $\{\eta < k(\beta) \mid k(\kappa)^{+\eta} \in k``(z \cap \kappa^{+\beta}) = k``(Z^\gamma_z \cap \beta)$. To prove the Lemma, it suffices to show that $k``(z \cap \kappa^{+\beta}) \in k(A^*)$, i.e. for all $\alpha \in Z^\gamma_z \cap \beta$, $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\}$ is in the measure represented by the map $y \mapsto u^\beta_\alpha(y)$. Call this measure $w$ and note that it is a lifted version of $\bar{u}^\beta_\alpha(y)$. Also note that $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\} = k``(A_\alpha)$, which collapses to $\bar{A}_\alpha \in \bar{u}^\beta_\alpha(y)$. Thus, $\{x \in k(A_\alpha) \mid x \prec k``(z \cap \kappa^{+\beta})\} \in w$, completing the proof of the lemma. □

For $\beta < \theta$, let $X_\beta$ be the set of $y \in \mathcal{P}_\kappa(\kappa^{+\beta})$ satisfying all of the statements in Lemma 2.1. Fix $\eta < \theta$.

## 3. The main forcing

We now define our forcing, $\mathbb{P}_{\vec{U},\eta}$. Conditions of $\mathbb{P}_{\vec{U},\eta}$ are pairs $(a, A)$ satisfying the following requirements.

(1) $a$ and $A$ are functions, $\mathrm{dom}(a)$ is a finite subset of $\theta \setminus \eta$, and $\mathrm{dom}(A) = \theta \setminus (\mathrm{dom}(a) \cup \eta)$.
(2) For all $\beta \in \mathrm{dom}(a)$, $a(\beta) \in X_\beta$.
(3) For all $\alpha < \beta$, both in $\mathrm{dom}(a)$, $a(\alpha) \prec a(\beta)$ and $\alpha \in Z^\beta_{a(\beta)}$.
(4) For all $\alpha \in \theta \setminus \max(\mathrm{dom}(a))$ (or, if $\mathrm{dom}(a) = \emptyset$, for all $\alpha \in \mathrm{dom}(A)$), $A(\alpha) \in U_\alpha$.
(5) For all $\alpha \in \mathrm{dom}(A) \cap \max(\mathrm{dom}(a))$, if $\beta = \min(\mathrm{dom}(a) \setminus \alpha)$, then $A(\alpha) \in u^\beta_\alpha(a(\beta))$ if $\alpha \in Z^\beta_{a(\beta)}$ and $A(\alpha) = \emptyset$ if $\alpha \notin Z^\beta_{a(\beta)}$.
(6) For all $\beta \in \mathrm{dom}(A)$ such that $A(\beta) \neq \emptyset$ and $\mathrm{dom}(a) \cap \beta \neq \emptyset$, if $\alpha = \max(\mathrm{dom}(a) \cap \beta)$, then, for all $y \in A(\beta)$, $a(\alpha) \prec y$ and $\alpha \in Z^\beta_y$.

If $(a, A), (b, B) \in \mathbb{P}_{\vec{U},\eta}$, then $(b, B) \leq (a, A)$ iff the following requirements hold.

(1) $b \supseteq a$.
(2) For all $\alpha \in \mathrm{dom}(b) \setminus \mathrm{dom}(a)$, $b(\alpha) \in A(\alpha)$.
(3) For all $\alpha \in \mathrm{dom}(B)$, $B(\alpha) \subseteq A(\alpha)$.

$(b, B) \leq^* (a, A)$ if $(b, B) \leq (a, A)$ and $b = a$.

**Remark** In our arguments, for notational simplicity we will typically assume $\eta = 0$ and then denote $\mathbb{P}_{\vec{U},\eta}$ as $\mathbb{P}_{\vec{U}}$. Everything proved about $\mathbb{P}_{\vec{U}}$ can be proved for a general $\mathbb{P}_{\vec{U},\eta}$ in the same way by making the obvious changes. The reason we introduce the more general forcing is to be able to properly state the factorization lemma.

In what follows, let $\mathbb{P}$ denote $\mathbb{P}_{\vec{U}}$. For any condition $p = (a, A) \in \mathbb{P}$, we often denote $(a, A)$ as $(a^p, A^p)$ and let $\gamma^p = \max(\mathrm{dom}(a^p))$. We refer to $a^p$ as the *stem* of $p$. Note that, if $p, q \in \mathbb{P}$ and $a^p = a^q$, then $p$ and $q$ are compatible. If $a$ is a non-empty stem, then let $\gamma^a$ denote $\max(\mathrm{dom}(a))$, and let $a^- = a \restriction \gamma^a$. Suppose $a$ is a stem, $\alpha < \theta$, and $x \in X_\alpha$. Suppose moreover that either $a$ is empty or $\gamma^a < \alpha$, $a(\gamma^a) \prec x$, and $\gamma^a \in Z^\alpha_x$. Then $a^\frown(\alpha, x)$ is a stem and $(a^\frown(\alpha, x))^- = a$. If $p \in \mathbb{P}$ and $b$ is a stem, then $b$ is *possible* for $p$ if there is $q \leq p$ with $a^q = b$. If $p \in \mathbb{P}$ and $b$ is possible for $p$, then $b^\frown p$ is the maximal $q$ such that $q \leq p$ and $a^q = p$. Such a $q$ always exists.

**Lemma 3.1.** *Suppose $(a, A) \in \mathbb{P}$, $\beta \in \mathrm{dom}(A)$, and $A(\beta) \neq \emptyset$. Then there is $(b, B) \leq (a, A)$ such that $\beta \in \mathrm{dom}(b)$.*

*Proof.* Straightforward using Lemmas 2.1, 2.2, and 2.3. □

**Definition** Suppose that $G$ is a $\mathbb{P}$-generic over $V$. Let Let $C^{\mathrm{sc}}_G$ (sc for super compact) be the set of all points $x = a(\beta)$ where $\beta \in \mathrm{dom}(a)$ for some $p = (a, A)$ in the generic filter $G$, and $C_G = \{\kappa_x \mid x \in C^{\mathrm{sc}}_G\}$ be the generic Radin Club.

**Lemma 3.2.** *$C_G$ is club in $\kappa$ and the assignment $x \mapsto \kappa_x = x \cap \kappa$ is an increasing bijection.*

*Proof.* Straightforward by Lemma 3.1 and genericity. □

Suppose $\beta < \theta$ and $y \in X_\beta$. Let $\vec{U}_y = \langle \bar{u}^\beta_\alpha(y) \mid \alpha \in Z^\beta_y \rangle$. For $\xi < f^\beta_\beta(y)$, let $\alpha_\xi \in Z^\beta_y$ be such that $\mathrm{otp}(\alpha \cap Z^\beta_y) = \xi$. Then $\bar{u}^\beta_\alpha(y)$ is a measure on $\mathcal{P}_{\kappa_y}(\kappa_y^{+\xi})$. Let $V_\xi = \bar{u}^\beta_{\alpha_\xi}(y)$. Then $\vec{U}_y = \langle V_\xi \mid \xi < f^\beta_\beta(y) \rangle$, and we can define $\mathbb{P}_{\vec{U}_y}$ as above.

If $p \in \mathbb{P}$, then $\mathbb{P}/p = \{q \in \mathbb{P} \mid q \leq p\}$.

**Lemma 3.3.** *(Factorization Lemma) Let $p = (a, A) \in \mathbb{P}$. Suppose $a \neq \emptyset$, $\beta = \max(\mathrm{dom}(a))$, and $y = a(\beta)$. Then there is $p' \in \mathbb{P}_{\vec{U}_y}$ such that $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_y}/p' \times \mathbb{P}_{\vec{U},\beta+1}/(\emptyset, A \restriction (\beta, \theta))$.*

*Proof.* Let $\pi : y \to \mathrm{otp}(y)$ be the unique order-preserving bijection. Define $p' = (a', A') \in \mathbb{P}_{\vec{U}_y}$ as follows. For $\xi < f^\beta_\beta(y)$, let $\alpha_\xi \in Z^\beta_y$ be such that $\mathrm{otp}(\alpha_\xi \cap Z^\beta_y) = \xi$. Let $\mathrm{dom}(a') = \{\xi < f^\beta_\beta(y) \mid \alpha_\xi \in \mathrm{dom}(a)\}$ and, for $\xi \in \mathrm{dom}(a')$, let $a'(\xi) = \pi``a(\alpha_\xi)$. Then $\mathrm{dom}(A') = f^\beta_\beta(y) \setminus \mathrm{dom}(a')$. If $\xi \in \mathrm{dom}(A')$, let $A'(\xi) = \{\pi``x \mid x \in A(\alpha_\xi)\}$. It is straightforward to verify that $p'$ thus defined is in $\mathbb{P}_{\vec{U}_y}$ and that $\mathbb{P}/p \cong \mathbb{P}_{\vec{U}_y}/p' \times \mathbb{P}_{\vec{U},\beta+1}/(\emptyset, A \restriction (\beta, \theta))$. □

By repeatedly applying the Factorization Lemma, standard arguments (see, e.g. [11]) allow us to assume we are working below a condition of the form $(\emptyset, A)$ when proving the following lemmas about $\mathbb{P}$.

**Lemma 3.4.** $(\mathbb{P}, \leq, \leq^*)$ *satisfies the Prikry property, i.e. if $\varphi$ is a statement in the forcing language and $p \in \mathbb{P}$, then there is $q \leq^* p$ such that $q \parallel \varphi$.*

*Proof.* The proofs of this and the next few lemmas are similar to those for the classical Radin forcing, which can be found in [11]. Fix $\varphi$ in the forcing language and $p \in \mathbb{P}$. By the Factorization Lemma, it suffices to let $p = (\emptyset, A)$ for some $A$. Let $a$ be a stem possible for $p$, and let $\alpha \in \theta \setminus (\gamma^a + 1)$. Let $Y_{a,\alpha} = \{x \in A(\alpha) \mid a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x\}$. Note that $Y_{a,\alpha} \in U_\alpha$. Let $Y^0_{a,\alpha} = \{x \in Y_{a,\alpha} \mid$ for some $B$, $(a^\frown(\alpha, x), B) \Vdash \varphi\}$, $Y^1_{a,\alpha} = \{x \in Y_{a,\alpha} \mid$ for some $B$, $(a^\frown(\alpha, x), B) \Vdash \neg\varphi\}$, and $Y^2_{a,\alpha} = Y_{a,\alpha} \setminus (Y^0_{a,\alpha} \cup Y^1_{a,\alpha})$. Fix $i(a, \alpha) < 3$ such that $Y^{i(a,\alpha)}_{a,\alpha} \in U_\alpha$, and let $Y^*_{a,\alpha} = Y^{i(a,\alpha)}_{a,\alpha}$.

For $\alpha < \theta$, let $B(\alpha)$ be the set of $x \in A(\alpha)$ such that, for every stem $a$ possible for $p$ such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{a,\alpha}$. We claim that $B(\alpha) \in U_\alpha$. Let $j = j_\alpha$. It suffices to show that $j``\kappa^{+\alpha} \in j(B(\alpha))$. Let $j(\langle Y^*_{a,\alpha} \mid a$ is a stem possible for $p$ and $\gamma^a < \alpha\rangle) = \langle W^*_{a,j(\alpha)} \mid a$ is a stem in $j(\mathbb{P})$ possible for $j(p)$ and $\gamma^a < j(\alpha)\rangle$. Suppose that, in $j(\mathbb{P})$, $a$ is a stem possible for $j(p)$ such that $a(\gamma^a) \prec j``\kappa^{+\alpha}$ and $\gamma^a \in j``\alpha$. Then there is a stem $\bar{a}$ possible for $p$ such that $j(\bar{a}) = a$. Then $W^*_{a,j(\alpha)} = j(Y^*_{\bar{a},\alpha})$, so, as $Y^*_{\bar{a},\alpha} \in U_\alpha$, $j``\kappa^{+\alpha} \in W^*_{a,j(\alpha)}$ and hence $j``\kappa^{+\alpha} \in j(B(\alpha))$. Thus, $(\emptyset, B) \in \mathbb{P}$ and $(\emptyset, B) \leq^* p$.

Suppose for sake of contradiction that no direct extension of $(\emptyset, B)$ decides $\varphi$. Find $(a, B^*) \leq (\emptyset, B)$ deciding $\varphi$ with $|a|$ minimal. Without loss of generality, suppose $(a, B^*) \Vdash \varphi$. Because of our assumption that no direct extension of $(\emptyset, B)$ decides $\varphi$, $a$ is non-empty. Let $b = a^-$ and $\gamma = \gamma^a$. By our construction of $B$, we have $a(\gamma) \in Y^0_{b,\gamma}$, and, for any $x \in B(\gamma)$ such that $b^\frown(\gamma, x)$ is a stem, there is $\hat{B}_x$ such that $(b^\frown(\gamma, x), \hat{B}_x) \Vdash \varphi$. Let $p^* = b^\frown(\emptyset, B) = (b, B^*)$. We will find a direct extension $(b, F)$ of $p^*$ forcing $\varphi$, thus contradicting the minimality of $|a|$.

We first define $F \restriction \gamma^b$ (if $b = \emptyset$, then there is nothing to do here). Since there are fewer than $\kappa$ possibilities for $F \restriction \gamma^b$ and $U_\gamma$ is $\kappa$-complete, we may fix a function $F^*$ on $\gamma^b \setminus \mathrm{dom}(b)$ such that $B_0(\gamma) := \{x \in B(\gamma) \mid b^\frown(\gamma, x)$ is a stem and $\hat{B}_x \restriction \gamma^b = F^*\} \in U_\gamma$. Then, for all $\alpha \in \gamma^b \setminus \mathrm{dom}(b)$, let $F(\alpha) = F^*(\alpha) \cap B^*(\alpha)$. We next define $F$ on the interval $(\gamma^b, \gamma)$ (or on all of $\gamma$, if $b = \emptyset$). If $\alpha \in (\gamma^b, \gamma)$, $x \in B_0(\gamma)$, and $\alpha \in Z^\gamma_x$, note that $\hat{B}_x(\alpha) \in u^\gamma_\alpha(x)$. Let $\bar{B}_x(\alpha)$ be the collapsed version of $\hat{B}_x(\alpha)$. Then $\bar{B}_x(\alpha) \in \bar{u}^\gamma_\alpha(x)$. Let $F^*(\alpha)$ be the set in $U_\alpha$ represented by the function $x \mapsto \bar{B}_x(\alpha)$ in the ultrapower by $U_\gamma$, and let $F(\alpha) = F^*(\alpha) \cap B^*(\alpha)$. Let $F(\gamma)$ be the set of $x \in B_0(\gamma) \cap B^*(\gamma)$ such that, for all $\alpha \in Z^\gamma_x \setminus \gamma^b$, $\{y \in F(\alpha) \mid y \prec x\} = \hat{B}_x(\alpha)$. We claim that $F(\gamma) \in U_\gamma$. To see this, let $j = j_\gamma$. Note that the function $x \mapsto \hat{B}_x(\alpha)$ represents $\{j``y \mid y \in F(\alpha)\}$, which is equal to $\{z \in j(F(\alpha)) \mid z \prec j``\kappa^{+\gamma}\}$. Thus, $j``\kappa^{+\gamma} \in j(F(\gamma))$, so $F(\gamma) \in U_\gamma$. We finally define $F$ on $(\gamma, \theta)$. If $\alpha \in (\gamma, \theta)$, let $F(\alpha)$ be the set of $y \in B^*(\alpha)$ such that $\gamma \in Z^\alpha_y$ and, for all $x \in F(\gamma)$ such that $x \prec y$, $y \in \hat{B}_x$. By now-familiar arguments, $F(\alpha) \in U_\alpha$. Notice that, by our construction, if $(c, H) \leq (b, F)$ and $\gamma \in \mathrm{dom}(c)$, then $(c, H) \leq (b^\frown(\gamma, c(\gamma)), \hat{B}_{c(\gamma)})$.

Now suppose for sake of contradiction that $(b, F) \not\Vdash \varphi$. Find $(c, H) \leq (b, F)$ such that $(c, H) \Vdash \neg\varphi$. If $\gamma \in \mathrm{dom}(c)$, then $(c, H) \leq (b^\frown(\gamma, c(\gamma)), \hat{B}_{c(\gamma)}) \Vdash \varphi$, which is a contradiction. Thus, suppose $\gamma \notin \mathrm{dom}(c)$. By our choice of $F(\alpha)$ for $\alpha \in (\gamma, \theta)$ (namely, our requirement that $\gamma \in Z^\alpha_y$ for all $y \in F(\alpha)$), it must be the case that $H(\gamma) \neq \emptyset$. But then $(c, H)$ can be extended further to a condition $(c', H')$ such that $\gamma \in \mathrm{dom}(c')$, and this again gives a contradiction. $\square$

**Definition** A tree $T \subseteq [\bigcup_{\alpha<\theta}(\{\alpha\} \times \mathcal{P}_\kappa(\kappa^{+\alpha}))]^{\leq n}$ is *fat* if the following conditions hold.

1. For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i_0 < i_1 \leq k$, we have $\alpha_{i_0} < \alpha_{i_1}$ and $x_{i_0} \prec x_{i_1}$.
2. For all $t \in T$ with $\mathrm{lh}(t) < n$, there is $\alpha_t < \theta$ such that:
   (a) for all $(\beta, y)$ such that $t^\frown(\beta, y) \in T$, $\beta = \alpha_t$;
   (b) $\{x \mid t^\frown(\alpha_t, x) \in T\} \in U_{\alpha_t}$.

If $T$ is as in the previous definition, then $n$ is said to be the *height* of $T$.

**Definition** Suppose $T$ is a fat tree, $\alpha < \theta$, and $x \in X_\alpha$. $T$ is *fat above* $(\alpha, x)$ *if, for all* $\langle(\alpha_i, x_i) \mid i \leq k\rangle$ *and all* $i \leq k$*, we have* $\alpha < \alpha_i$ *and* $x \prec x_i$.

**Definition** Suppose $\gamma < \theta$ and $z \in X_\gamma$. A tree $T \subseteq [\bigcup_{\alpha<\theta}(\{\alpha\} \times \mathcal{P}_\kappa(\kappa^{+\alpha}))]^{\leq n}$ is *fat below* $(\gamma, z)$ if the following conditions hold.

1. For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i \leq k$, we have $\alpha_i \in Z^\gamma_z$ and $x_i \prec z$.
2. For all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and all $i_0 < i_1 \leq k$, we have $\alpha_{i_0} < \alpha_{i_1}$ and $x_{i_0} \prec x_{i_1}$.
3. For all $t \in T$ with $\mathrm{lh}(t) < n$, there is $\alpha_t \in Z^\gamma_z$ such that:
   (a) for all $(\beta, y)$ such that $t^\frown(\beta, y) \in T$, $\beta = \alpha_t$;
   (b) $\{x \mid t^\frown(\alpha_t, x) \in T\} \in u^\gamma_{\alpha_t}(z)$.

Suppose $T$ is a fat tree, $\gamma < \theta$, and $z \in \mathcal{P}_\kappa(\kappa^{+\gamma})$. $T \restriction (\gamma, z)$ is the subtree of $T$ consisting of all $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ such that, for all $i \leq k$, $\alpha_i \in Z^\gamma_z$ and $x_i \prec z$.

If $T$ is a fat tree, let $\gamma_T = \sup(\{\alpha \mid$ for some $\langle(\alpha_i, x_i) \mid i \leq k\rangle \in T$ and $i \leq k$, $\alpha = \alpha_i\})$. Note that, if $\theta$ is weakly inaccessible and $|\mathcal{P}_\kappa(\kappa^{+\alpha})| < \theta$ for all $\alpha < \theta$, we have $\gamma_T < \theta$.

Suppose that $S$ is a set of stems and, for all $a \in S$, $T_a$ is a fat tree above $(\gamma^a, a(\gamma^a))$. For $\gamma < \theta$, let $S_{<\gamma} = \{a \in S \mid \gamma^a < \gamma\}$. Let $C = \{\gamma < \theta \mid$ for all $a \in S_{<\gamma}$, $\gamma_{T_a} < \gamma\}$. Note that $C$ is club in $\theta$. For all $\gamma \in C$, let $Y_\gamma$ be the set of $z \in X_\gamma$ such that, for all $a \in S_{<\gamma}$ such that $a(\gamma^a) \prec z$ and $\gamma^a \in Z^\gamma_z$, $T_a \restriction (\gamma, z)$ is fat below $(\gamma, z)$. We claim that $Y_\gamma \in U_\gamma$. To see this, let $j = j_\gamma$, and note first that $\{a \in j(S_{<\gamma}) \mid a \prec j``\kappa^{+\gamma}$ and $\gamma^a \in j``\gamma\} = j``S_{<\gamma}$ and second that, for all $\bar{a} \in S_{<\gamma}$, $j(T_{\bar{a}}) \restriction (j(\gamma), j``\kappa^{+\gamma}) = j``T_{\bar{a}}$, which is fat below $(j(\gamma), j``\kappa^{+\gamma})$. Thus, $j``\kappa^{+\gamma} \in j(Y_\gamma)$, so $Y_\gamma \in U_\gamma$.

**Lemma 3.5.** *Suppose $p = (a, A) \in \mathbb{P}$ and $D \subseteq \mathbb{P}$ is a dense open set. Suppose $a = \{(\alpha_i, x_i) \mid i < k\}$ is such that, for all $i_0 < i_1 < k$, $\alpha_{i_0} < \alpha_{i_1}$. There are trees $\langle T_i \mid i \leq k\rangle$ and natural numbers $\langle n_i \mid i \leq k\rangle$ such that the following hold.*

1. *For all $i \leq k$, $T_i$ is a tree of height $n_i$.*
2. *If $k > 0$, then $T_0$ is fat below $(\alpha_0, x_0)$ and, for all $0 < i < k$, $T_i$ is fat below $(\alpha_i, x_i)$ and above $(\alpha_{i-1}, x_{i-1})$.*
3. *$T_k$ is fat, and, if $k > 0$, it is above $(\alpha_{k-1}, x_{k-1})$.*
4. *Suppose that, for all $i \leq k$, $\langle(\beta^i_\ell, y^i_\ell) \mid \ell < n_i\rangle$ is a maximal element of $T_i$. Then, if $b = a \cup \{(\beta^i_\ell, y^i_\ell) \mid i \leq k, \ell \leq n_i\}$, there is $B$ such that $(b, B) \leq (a, A)$ and $(b, B) \in D$.*

*Proof.* By the Factorization Lemma, it again suffices to consider $p$ of the form $(\emptyset, A)$. We thus need to find a single fat tree $T$. We inductively construct a decreasing sequence of conditions $\langle(\emptyset, A_n) \mid n < \omega\rangle$. Intuitively, $A_n$ will take care of extensions $(b, B) \leq (\emptyset, A)$ such that $|b| = n$. We explicitly go through the first few steps of the construction.

Let $A_0 = A$. If there is a direct extension of $(\emptyset, A)$ in $D$, then we are done by setting $T = \{\emptyset\}$. Thus, suppose there is no such direct extension. For every stem $a$ possible for $(\emptyset, A_0)$ and every $\alpha \in (\gamma^a, \theta)$, let $Y_{0,a,\alpha} = \{x \in A_0(\alpha) \mid a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x\}$. Let $Y^0_{0,a,\alpha} = \{x \in Y_{0,a,\alpha} \mid$ for some $B$, $(a^\frown(\alpha, x), B) \in D\}$, and let $Y^1_{0,a,\alpha} = Y_{0,a,\alpha} \setminus Y^0_{0,a,\alpha}$. Find $i(0, a, \alpha) < 2$ such that $Y^{i(0,a,\alpha)}_{0,a,\alpha} \in U_\alpha$, and let $Y^*_{0,a,\alpha} = Y^{i(0,a,\alpha)}_{0,a,\alpha}$. For $\alpha < \theta$, let $A_1(\alpha)$ be the set of $x \in A_0(\alpha)$ such that, for all stems $a$ possible for $(\emptyset, A_0)$ such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{0,a,\alpha}$. As in the proof of Lemma 3.4, $A_1(\alpha) \in U_\alpha$ for all $\alpha < \theta$, so $(\emptyset, A_1) \leq^* (\emptyset, A_0)$. Note that $(\emptyset, A_1)$ satisfies the following property, which we denote $(*)_1$:

> Suppose $q = (a^\frown(\alpha, x), B) \leq (\emptyset, A_1)$ and $q \in D$. Then, for every $y \in A_1(\alpha)$ such that $a(\gamma^a) \prec y$ and $\gamma^a \in Z^\alpha_y$, there is $B_y$ such that $(a^\frown(\alpha, y), B_y) \in D$.

Now suppose there is a stem $a = \{(\alpha, x)\}$ possible for $(\emptyset, A_1)$ and a $B$ such that $(a, B) \in D$. We can then define a fat tree $T$ of height 1 whose maximal elements are all $\langle(\alpha, x)\rangle$ such that $x \in A_1(\alpha)$. We are then done, as $T$ easily satisfies the requirements of the lemma. Thus, suppose there is no such $a$ and proceed to define $(\emptyset, A_2)$ as follows.

For every stem $a$ possible for $(\emptyset, A_1)$ and every $\alpha \in (\gamma^a, \theta)$, let $Y_{1,a,\alpha} = \{x \in A_1(\alpha) \mid a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x\}$. Let $Y^0_{1,a,\alpha}$ be the set of all $x \in Y_{1,a,\alpha}$ such that there are $\beta^\alpha_x \in (\alpha, \theta)$ and $W^\alpha_x \in U_{\beta^\alpha_x}$ such that, for all $y \in W^\alpha_x$:

- $x \prec y$ and $\alpha \in Z^{\beta^\alpha_x}_y$;
- there is $B$ such that $(a^\frown(\alpha, x)^\frown(\beta^\alpha_x, y), B) \in D$.

Let $Y^1_{1,a,\alpha} = Y_{1,a,\alpha} \setminus Y^0_{1,a,\alpha}$. Find $i(1, a, \alpha) < 2$ such that $Y^{i(1,a,\alpha)}_{1,a,\alpha} \in U_\alpha$, and let $Y^*_{1,a,\alpha} = Y^{i(1,a,\alpha)}_{1,a,\alpha}$. For $\alpha < \theta$, let $A_2(\alpha)$ be the set of $x \in A_1(\alpha)$ such that, for all stems $a$ possible for $(\emptyset, A_1)$ such that $a(\gamma^a) \prec x$ and $\gamma^a \in Z^\alpha_x$, $x \in Y^*_{1,a,\alpha}$. Then $(\emptyset, A_2) \leq^* (\emptyset, A_1)$, and $(\emptyset, A_2)$ satisfies the property $(*)_2$:

> Suppose $q = (a^\frown(\alpha, x)^\frown(\beta, y), B) \leq (\emptyset, A_2)$ and $q \in D$. Then, for every $x' \in A_2(\alpha)$ such that $a(\gamma^a) \prec x'$ and $\gamma^a \in Z^\alpha_{x'}$, there is $\beta^\alpha_{x'} \in (\alpha, \theta)$ and $W^\alpha_{x'} \in U_{\beta^\alpha_{x'}}$ such that, for all $y' \in W^\alpha_{x'}$, there is $B'$ such that $(a^\frown(\alpha, x')^\frown(\beta^\alpha_{x'}, y'), B') \in D$.

Suppose there is a stem $a = \{(\alpha, x)^\frown(\beta, y)\}$ possible for $(\emptyset, A_2)$ and a $B$ such that $(a, B) \in D$. Using $(*)_2$, we can define a tree $T$ of height 2 whose maximal elements are all $\langle(\alpha, x')^\frown(\beta^\alpha_{x'}, y')\rangle$ such that $x' \in A_2(\alpha)$ and $y' \in W^\alpha_{x'}$. We are then done, as $T$ satisfies the requirements of the lemma. If there is no such stem $a$, then continue in the same manner.

In this way, we can construct $A_n$ such that, if there is a stem $a$ possible for $(\emptyset, A_n)$ with $|a| = n$ and a $B$ such that $(a, B) \in D$, then there is a fat tree of height $n$ as desired. For $\alpha < \theta$, let $A_\infty(\alpha) = \bigcap_{n<\omega} A_n(\alpha)$. For all $n < \omega$, $(\emptyset, A_\infty) \leq^* (\emptyset, A_n)$. Find $(a, B) \leq (\emptyset, A_\infty)$ such that $(a, B) \in D$. Let $n^* = |a|$. Then $a$ is possible for $(\emptyset, A_{n^*})$, so there is a fat tree of height $n^*$ as required by the lemma. □

**Theorem 3.6.** *If $\theta$ is weakly inaccessible and $|\mathcal{P}_\kappa(\kappa^{+\alpha})| < \theta$ for all $\alpha < \theta$, then $\kappa$ remains inaccessible in $V^{\mathbb{P}}$.*

*Proof.* It suffices to prove that $\kappa$ remains regular. Let $p = (a, A) \in \mathbb{P}$, let $\delta < \kappa$, and suppose $\dot{f}$ is a $\mathbb{P}$-name forced by $p$ to be a function from $\delta$ to $\kappa$. We will find $q \leq p$ forcing the range of $\dot{f}$ to be bounded below $\kappa$.

For all $\xi < \delta$, let $D_\xi$ be the set of $(b, B) \in \mathbb{P}$ such that $(b, B) \Vdash$ "$\dot{f}(\xi) < \kappa_{b(\gamma^b)}$". Each $D_\xi$ is a dense, open subset of $\mathbb{P}$. For $\xi < \delta$, let $S_\xi$ be the set of stems $b$ such that, for some $B$, $(b, B) \leq p$ and $(b, B) \in D_\xi$. For all $b \in S_\xi$, fix a $B^\xi_b$ witnessing this. For each $\beta \in (\gamma^a, \theta)$, let $A^*(\beta)$ be the set of $y \in A(\beta)$ such that, for all $\xi < \delta$ and all $b \in S_\xi$ such that $b(\gamma^b) \prec y$ and $\gamma^b \in Z^\beta_y$, $y \in B^\xi_b(\beta)$. For $\beta \in \mathrm{dom}(A) \cap \gamma^a$, let $A^*(\beta) = A(\beta)$. Then $(a, A^*) \leq (a, A)$. Let $T$ be the set of stems possible for $(a, A^*)$. For $\gamma < \theta$, let $T_{<\gamma} = \{c \in T \mid \gamma^c < \gamma\}$. For all $c \in T$, let $p_c = c^\frown(a, A^*)$. For all $c \in T$ and $\xi < \delta$, apply Lemma 3.5 to $p_c$ and $D_\xi$ to obtain a sequence of trees $\langle T_{c,\xi,i} \mid i \leq |c|\rangle$. Let $C$ be the set of $\gamma < \theta$ such that, for all $c \in T_{<\gamma}$ and all $\xi < \delta$, $\gamma_{T_{c,\xi,|c|}} < \gamma$. $C$ is club in $\theta$. Fix $\gamma \in C \setminus (\gamma^a + 1)$. By the discussion preceding Lemma 3.5, choose $z \in A^*(\gamma)$ such that, for all $c \in T_{<\gamma}$ such that $c(\gamma^c) \prec z$ and $\gamma^c \in Z^\gamma_z$ and for all $\xi < \delta$, $T_{c,\xi,|c|} \restriction (\gamma, z)$ is fat below $(\gamma, z)$. Then $q = (a^\frown(\gamma, z), A^{**}) \leq (a, A^*)$, where $A^{**}(\alpha) = A^*(\alpha)$ for all $\alpha \in \mathrm{dom}(A^*) \cap \gamma^a$ and all $\alpha \in (\gamma, \theta)$, and $A^{**}(\alpha) = \{x \in A^*(\alpha) \mid x \prec z\}$ for all $\alpha \in (\gamma^a, \gamma)$.

We claim that $q$ forces the range of $\dot{f}$ to be bounded below $\kappa_z$. Suppose for sake of contradiction that there is $\xi < \delta$ and $r \leq q$ such that $r \Vdash$ "$\dot{f}(\xi) \geq \kappa_z$." Let $r = (d, F)$, and let $c = \{(\alpha, x) \mid \alpha < \gamma\}$. Then $c \in T_{<\gamma}$, $c(\gamma^c) \prec z$, and $\gamma^c \in Z^\gamma_z$, so $T_{c,\xi,|c|} \restriction (\gamma, z)$ is fat below $(\gamma, z)$. Suppose that, for all $i \leq |c|$, $n_i$ is the height of $T_{c,\xi,i}$. Then, for all $i \leq |c|$, we can find maximal elements $\langle(\beta^i_\ell, y^i_\ell) \mid \ell < n_i\rangle$ of $T_{c,\xi,i}$ such that, letting $c' = c \cup \{(\beta^i_\ell, y^i_\ell) \mid i \leq |c|, \ell < n_i\}$, $c' \cup d$ is possible for $r$. In particular, for all $(\alpha, x) \in c'$, $x \prec z$ and $\alpha \in Z^\gamma_z$, and so $\kappa_x < \kappa_z$. Also, there is $B'$ such that $(c', B') \in D_\xi$ and, by our construction of $A^*$, we may assume that, for $\alpha \in (\gamma^{c'}, \theta)$, $B'(\alpha) = A^*(\alpha)$. All of this together means that $(c', B')$ and $r$ are compatible. However, as $(c', B') \in D_\xi$, $(c', B') \Vdash$ "$\dot{f}(\xi) < \kappa_{c'(\gamma^{c'})} < \kappa_z$," contradicting the assumption that $r \Vdash$ "$\dot{f}(\xi) \geq \kappa_z$." □

## 4. Approachability

In this section we characterize precisely which successors $\nu^+$ for $\nu \in C$ have reflection properties.

**Notation** Suppose that $\beta < \theta$ and that $y \in X_\beta$ and $p$ is a condition with $a^p(\beta) = y$. For notational simplicity we write $o(y)$ for $f^\beta_\beta(y) = otp(Z^\beta_y)$. Note that $o(y) < \nu^{+\theta(\kappa_y)}$ where $\theta(\kappa_y)$ is the next inaccessible cardinal above $\kappa_y$.

**Lemma 4.1.** *$\mathbb{P}_{\vec{U}_y}$ as defined above Lemma 3.3 has the $\kappa_y^{+o(y)+1}$-Knaster property.*

It is not hard to see that there are just $\kappa_y^{+o(y)}$ many possible $a$-parts of conditions in this poset and that conditions with the same $a$-part are compatible.

**Lemma 4.2.** *Suppose that we have $\beta < \theta$, $y \in X_\beta$, $y' \in X_{\beta+1}$ and $p$ is a condition such that $a^p(\beta) = y$ and $a^p(\beta + 1) = y'$. If $\mu$ is an ordinal of cofinality $\delta$ with $\kappa_y^{o(y)+1} \leq \delta < \kappa_{y'}$ and $\dot{C}$ is a $\mathbb{P}$-name for a club subset of $\mu$, then there are $p' \leq^* p$ and $D \subseteq \mu$ club such that $p'$ forces $D \subseteq \dot{C}$.*

*Proof.* First we show that it is enough to consider $\delta = \mu$. Assume for the moment that $\delta < \mu$. Let $\pi : \delta \to \mu$ be an increasing continuous and cofinal function. Now by passing to a name for a subset of $\dot{C}$ we can assume that it is forced that $\dot{C}$ is a subset of the range of $\pi$. Now a condition will force that there is a ground model contained in $\dot{C}$ if and only if there is a ground model club contained in $\pi^{-1}\dot{C}$.

So we may assume that $\delta = \mu$. By Lemma 3.3 and Lemma 3.4, there is a direct extension $p'$ of $p$ which forces $\dot{C}$ to be in the extension by $\mathbb{P}_{\vec{U}_y}$. Now by a standard argument using the $\kappa_y^{+o(y)+1}$-cc of $\mathbb{P}_{\vec{U}_y}$, there is a club $D \subseteq \mu$ in $V$ such that $p'$ forces $D \subseteq \dot{C}$. □

We use the above lemma to show that the approachability property fails at certain points along our Radin club.

**Lemma 4.3.** *Suppose that $\beta$ is a limit ordinal with $\mathrm{cf}^V(\beta) < \kappa$ and $p$ is a condition such that $a^p(\beta) = y$, then $p$ forces that $\kappa_y^+ \notin I[\kappa_y^+]$.*

*Proof.* Assume for a contradiction that (some extension of) $p$ forces $\kappa_y^+ \in I[\kappa_y^+]$. Let $\langle \dot{z}_\gamma \mid \gamma < \kappa_y^{+o(y)+1} \rangle$ be a name for the approachability witness. We can assume that the order type of each $\dot{z}_\gamma$ is forced to be less than $\kappa_y$.

Let $j : V \to M$ witness that $\kappa_y$ is $\kappa_y^{+o(y)+1}$-supercompact using an ultrapower which projects to $\bar{u}^\beta_\alpha(y)$ for some $\alpha \in Z^\beta_y$. In particular if we consider $j(\mathbb{P}_{\vec{U}_y})$, then $\hat{y} = j``\kappa_y^{+f^\beta_\alpha(\kappa_y)}$ is a possible value for the $j(a)(j(f^\beta_\alpha)(\kappa_y))$.

Let $\hat{p} \leq j(p)$ be a condition in $j(\mathbb{P}_{\vec{U}_y})$ such that $a^{\hat{p}}(f^\beta_\alpha(\kappa_y)) = \hat{y}$ and $a^{\hat{p}}(f^\beta_\alpha(\kappa_y) + 1) = \hat{y}'$ for some $\hat{y}'$. We set $\mu = \sup j``\kappa_y^{+o(y)+1}$ and $\delta = \mathrm{cf}(\mu)$. We have that $\hat{p}$ forces that $\mu$ is approachable with respect to $j(\dot{z}_\gamma \mid \gamma < \kappa_y^{o(y)+1}\rangle)$. So there is a name $\dot{C}$ for a club subset of $\mu$ such that for all $\gamma < \mu$, $\dot{C} \cap \gamma$ is enumerated as $j(\dot{z})_{\gamma'}$ for some $\gamma' < \mu$.

By the previous lemma there are a club $D \subseteq \mu$ in $M$ and a direct extension $\hat{p}'$ of $\hat{p}$ such that $\hat{p}'$ forces $D \subseteq \dot{C}$. Let $E = \{\gamma < \kappa_y^{o(y)+1} \mid j(\gamma) \in D\}$. It is straightforward to see that $E$ is $< \kappa_y$-club in $\kappa_y^{o(y)+1}$. Let $\gamma^*$ be the $\kappa_y^{o(y)}$th element in some enumeration of $E$. We can assume that there is an index $\hat{\gamma}$ such that $\hat{p}'$ forces that $\dot{C} \cap j(\gamma^*)$ is enumerated before stage $j(\hat{\gamma})$.

Note that $o(y) < \kappa_y$ since $\mathrm{cf}(\beta) < \kappa_y$. Now if $x \subseteq E \cap \gamma^*$ has ordertype at most $o(y)$, then there is a condition $p_x$ which forces $x \subseteq \dot{z}_\gamma$ for some $\gamma < \hat{\gamma}$. To see this notice that $j$ of this statement is witnessed by $\hat{p}'$.

By the chain condition of $\mathbb{P}_{\vec{U}_y}$, we can find a condition which forces that for $\kappa_y^{+o(y)+1}$ many $x$, $p_x$ is in the generic. This is impossible, since each $\dot{z}_\gamma$ is forced to have order type less than $\kappa_y$ and hence in the extension $|\bigcup_{\gamma<\hat{\gamma}} \mathcal{P}(\dot{z}_\gamma)| \leq \kappa_y$. $\square$

Next we show that weak square holds at points taken from $X_\beta$ where $\mathrm{cf}(\beta) \geq \kappa$. To do so we need a claim about the cofinalities of points in the extension and a few definitions.

Suppose that $\nu = \kappa \cap x \in C$ with $x \in C_G^{\mathrm{sc}}$. Let $p = (a, A) \in G$ so that $x = a(\beta)$ for some $\beta \in \mathrm{dom}(a)$. By the definition of the sets $X_\beta$, $\beta < \theta$ and $\mathbb{P}$, we have that

- $\beta$ is a limit ordinal if and only of $o(x)$ is,
- If $\beta$ is limit then $(\nu^+)^{V[G]} = (\nu^{+o(x)+1})^V$,
- $\mathrm{cf}(\beta) \geq \kappa$ if and only if $o(x) \geq \nu$.

**Lemma 4.4.** *If $\beta$ is limit and $\mathrm{cf}(\beta) \geq \kappa$, then $\nu$ and $\nu^{+o(x)}$ change their cofinality to $\omega$ in $V[G]$.*

*Proof.* As the forcing $\mathbb{P}/(\beta+1)$ does not add new subsets to $\nu^{\theta(\nu)}$ it is sufficient to focus on the forcing $\mathbb{P}_{\vec{U}_x}$ which adds a Radin club to $\nu$.

For notational simplicity suppose that $\nu = \kappa$ and then $\mathbb{P}_{U_{\vec{x}}} \cong \mathbb{P}_{\vec{U}}$ where $\vec{U} = \langle U_\alpha \mid \alpha < \beta \rangle$ where $\beta < \theta$ and $\mathrm{cf}(\beta) = \rho \geq \kappa$, and let us show that $\kappa$ and $\kappa^{+\beta}$ change their cofinality to $\omega$.

Choose an increasing continuous sequence $\vec{\beta} = \langle \beta_\alpha \mid \alpha < \rho \rangle$ cofinal in $\beta$. For every $y \in C_G^{\mathrm{sc}}$ let $\alpha(y) < \rho$ be the minimal $\alpha < \rho$ so that $y = a(\beta')$ for some $\beta' \leq \beta_\alpha$ and some $p = (a, A) \in G$.

Since $\theta$ is the first inaccessible cardinal above $\kappa$ then $\kappa^{+\rho} > \rho$. Let $\alpha_0 < \rho$ be the first ordinal so that $\rho < \kappa^{+\beta_{\alpha_0}}$. Let $\beta_0 = \beta_{\alpha_0}$
Note that for every $\beta'$, $\beta_0 \leq \beta' < \beta$ $Y_{\beta'} = \{x \in X_{\beta'} \mid \alpha(x) \in x\}$ belongs to $U_{\beta'} \subset \mathcal{P}(\mathcal{P}_\kappa(\kappa^{+\beta'}))$.

It follows that in $V[G]$ there is some $\nu_0 \in C$ such that for every $x \in C_G^{\mathrm{sc}}$, if $x = a(\beta')$ for some $\beta' \geq \beta_0$ and $\kappa_x = \kappa \cap x > \nu_0$, then $x \in Y_{\beta'}$. Let $y_0$ be the minimal $x \in C_G^{\mathrm{sc}}$ satisfying the above. Starting from $y_0$, we define a sequence $\vec{y} = \langle y_n \mid n < \omega \rangle \subset C_G^{\mathrm{sc}}$. For each $n < \omega$, let $y_{n+1}$ be the minimal $x$ above $y_n$ in $C_G^{\mathrm{sc}}$ so that $\alpha(x) > \sup(y_n \cap \rho) < \rho$. Let $\kappa_\omega = \cup_n \kappa_{y_n}$. We claim that $\kappa_\omega = \kappa$. Suppose otherwise. Then $\kappa_\omega = \kappa_y$ for some $y \in C_G^{\mathrm{sc}}$. Pick $p = (a, A) \in G$ and $\beta' \geq \beta_0$ so that $y = a(\beta')$. Let $\alpha = \alpha(y) < \rho$, then $\alpha \in y < \sup y \cap \rho$. Since $\langle \kappa_{y_n} \mid n < \omega \rangle$ are cofinal in $\kappa_\omega$ then $y \cap \rho = \cup_n (y_n \cap \rho)$, so there is some $m < \omega$ such that $\sup(y_m \cap \rho) > \alpha$. But $\alpha \geq \alpha(y_{m+1})$ which means that $\alpha(y_{m+1}) < \sup(y_m \cap \rho)$, contradicting the definition of the sequence $\vec{y}$.

It follows that $\kappa = \kappa_\omega$ so it changes its cofinality to $\omega$. The set $C_G^{\mathrm{sc}} \subset \mathcal{P}_\kappa(\kappa^{+\beta})$ is $\subseteq$ $-$cofinal in $\mathcal{P}_\kappa(\kappa^{+\beta})$. Since $\langle \kappa_n \mid n < \omega \rangle$ is cofinal in $\kappa$ then the sequence $\langle y_n \mid n < \omega \rangle$ is $\subset$ unbounded in $C_G^{\mathrm{sc}}$, thus $\kappa^{+\beta} = \cup_n y_n$. It follows that $\mathrm{cf}(\kappa^\beta) = \omega$ as each $y_n$ is bounded in $\kappa^{+\beta}$. $\square$

It follows easily that cardinals between $\nu$ and $\nu^{+o(\nu)}$ also change their cofinality to $\omega$. To show that weak square holds we need the definition of a partial square sequence.

**Definition** Let $\lambda < \delta$ be regular cardinals and $S \subseteq \delta \cap \mathrm{cof}(\lambda)$. We say that $S$ *has a partial square sequence* $\langle C_\gamma \mid \gamma \in S \rangle$ is a partial square sequence if

(1) for all $\gamma \in S$, $C_\gamma$ is club in $\gamma$ and $\mathrm{otp}(C_\gamma) = \lambda$ and

(2) for all $\gamma < \gamma^*$ from $S$ if $\beta$ is a limit point of $C_\gamma$ and $C_{\gamma^*}$, then $C_\gamma \cap \beta = C_{\gamma^*} \cap \beta$.

Next we need a theorem of Dzamonja and Shelah [5].

**Theorem 4.5.** *Let $\lambda$ be a regular cardinal and $\mu > \lambda$ be singular. If $\mathrm{cf}([\mu]^{\leq\lambda}, \subseteq) = \mu$, then $\mu^+ \cap \mathrm{cof}(\leq \lambda)$ is the union of $\mu$ many sets each of which carries a partial square.*

**Lemma 4.6.** *Suppose that $\beta < \theta$ with $\mathrm{cf}(\beta) \geq \kappa$ and $p$ is a condition where $a^p(\beta) = y$ for some $y$. Then $p$ forces $\square^*_{\kappa_y}$.*

*Proof.* For each regular $\lambda < \kappa_y$, we have that $(\kappa_y^{+o(y)+1})^{\leq\lambda} = \kappa_y^{+o(y)+1}$ using the supercompactness of $\kappa_y$ and hence $\mathrm{cf}([\kappa_y^{+o(y)+1}]^{\leq\lambda}, \subseteq) = \kappa_y^{+o(y)+1}$.

So we can write $\kappa_y^{+o(y)+1} \cap \mathrm{cof}(\leq \lambda)$ as the union of $\kappa_y^{+o(y)}$ sets which have partial squares. We call these sequences $\vec{C}^{\lambda,i}$ for $i < \kappa_y^{+o(y)}$.

By Lemma 4.4, in the extension we have that each cardinal in the interval $[\kappa_y, \kappa_y^{+o(y)}]$ changes its cofinality to $\omega$. So in the extension, we can write $\kappa_y^{+o(y)+1}$ as the disjoint union of $\kappa_y^{+o(y)+1} \cap \mathrm{cof}(< \kappa_y)$ which we call $T$ and a set $S$ of ordinals of countable cofinality.

We define a weak square sequence as follows. For $\gamma \in T$ we let $\mathcal{C}_\gamma = \{C^{\lambda,i}_{\gamma'} \mid \lambda < \kappa_y,\ i < \kappa_y^{+o(y)}$ and $\gamma$ is a limit point of $C^{\lambda,i}_{\gamma'}\}$. For $\gamma \in S$, we let $\mathcal{C}_\gamma = \{C\}$ where $C$ is some cofinal $\omega$-sequence in $\gamma$.

The coherence is obvious, so we just have to check that each $\mathcal{C}_\gamma$ is not too large. Suppose that there is $\gamma$ such that $|\mathcal{C}_\gamma| \geq \kappa_y^{+o(y)+1}$. Then by the Pigeonhole principle we can find two elements $C$ and $C'$ on which the indices $\lambda$ and $i$ are the same, but then we have that $C = C'$ by the coherence of the partial square sequence with indices $\lambda$ and $i$, a contradiction. $\square$

## 5. Iteration from below

In this section we describe how to iterate Sinapova's poset from [21] to obtain many singular cardinals $\nu$ where SCH fails and there are no special $\nu^+$ trees.

(1) Suppose that $\kappa$ is Mahlo and $\mathcal{U} = \langle U_{\alpha,\tau} \mid \alpha < \kappa, \tau < o^{\mathcal{U}}(\alpha)\rangle$ is a coherent sequence of supercompact measures so that for each $\alpha < \kappa$ and $\tau < o^{\mathcal{U}}(\alpha)$, $U_{\alpha,\tau}$ is a $\mathcal{P}_\alpha(\alpha^{+\tau})$ supercompact measure.
We further assume $o(\alpha) < \alpha$ for each $\alpha < \kappa$ and that for every $\tau < \kappa$, the set $\Delta_\tau = \{\alpha < \kappa \mid o^{\mathcal{U}}(\alpha) = \tau\}$ is stationary in $\kappa$.

(2) We define a Gitik-iteration $\mathbb{P}_\kappa = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha < \kappa\rangle$ of Sinapova's poset from [21].
By induction on $\alpha$, we define $\mathbb{P}_\alpha$ and $\mathbb{Q}_\alpha$ so that the following holds:
   (a) $\mathbb{Q}_\alpha$ will be nontrivial if and only if $o(\alpha) > 0$.
   (b) $(\mathbb{Q}_\alpha, \leq_\alpha, \leq^*_\alpha)$ is a Prikry type forcing notion and $\leq^*_\alpha$ is $< \alpha$ closed.
   (c) If $o(\alpha) > \alpha$ then a $\mathbb{Q}_\alpha$ generic filter assigns a sequence $\vec{x}^\alpha = \langle x^\alpha_i \mid i < o(\beta)\rangle$ which generic for the poset in [21] with respect to the measures $\langle U_{\beta,\tau} \mid \tau < o(\beta)\rangle$. In particular, if $o(\alpha)$ is a limit ordinal then $\alpha^{+o(\alpha)} = \bigcup_i x^\beta_i\ V[G_\alpha]$
   (d) We use the Gitik support from [9] to define $\mathbb{P}_\alpha$. Conditions $p \in \mathbb{P}_\alpha$ are of the form $\langle p_\beta \mid \beta \in g\rangle$ where $g \subset \alpha$ is an Easton support set.

Suppose that $\alpha \leq \kappa$ and that $\mathbb{P}_\alpha$ has been defined. Let $G \subset \mathbb{P}_\alpha$ be a generic filter. Suppose that $o(\alpha) = \lambda$. We define posets $\mathbb{Q}^\tau_\alpha$ for $\tau \leq \lambda$ by induction on $\tau$. $\mathbb{Q}^\tau_\alpha$ will add a generic sequence $\vec{x}^{\alpha,\tau} = \langle x^\alpha_i \mid i < \tau\rangle$ which is Prikry/Magidor generic for the sequence $\langle U_{\alpha,\gamma} \mid \gamma < \tau\rangle$.

(a) If $\lambda = 0$ then $\mathbb{Q}^0_\alpha = \mathbb{Q}_\alpha$ is trivial.

(b) Suppose $0 < \tau \leq \lambda$ and $\mathbb{Q}_{\alpha,\gamma}$ has been defined for every $\gamma < \tau$.
Suppose that $\gamma < \tau$ and $x \in \mathcal{P}_\alpha(\alpha^{+\gamma})$. Following Gitik's notations from [9], we say that $x$ is $\gamma-$coherent if $\alpha_x = \alpha \cap x < \alpha$ and $o(\alpha_x) = \gamma$. We say that $x$ is coherent if it is $\gamma-$coherent for some $\gamma$. For each such $x$, let $\pi_x : x \to \bar{x}$ be the transitive collapse.
Conditions $q \in \mathbb{Q}_{\alpha,\tau}$ will be of the form $\langle x, T\rangle$ where $x$ is $\gamma-$coherent for some $\gamma < \tau$ and $T$ is a tree whose splitting sets belong to ultrafilters of the form $U_{\alpha,\gamma'}(y)$ where $\gamma < \gamma' < \tau$ and $y$ is coherent.
More precisely, $T \subset [\mathcal{P}_\alpha(\alpha^{+\tau})]^{\leq\omega}$ satisfies that
- $stem(T) = \emptyset$,
- $\mathrm{Succ}_T(\emptyset) \in \bigcap\{U_{\alpha,\gamma'}(x) \mid \gamma < \gamma' < \tau\}$ and $x \prec y$ for every $y \in \mathrm{Succ}_T(\emptyset)$,
- For every $s = \langle x_0, \dots, x_k\rangle \in T$, $s$ is a $\prec$ increasing sequence of subsets in $\mathcal{P}_\alpha(\alpha^{+\tau})$ and the sequence $\langle o(\alpha_{x_0}), \dots, o(\alpha_{x_k})\rangle$ is an increasing sequence of ordinals below $\tau$.
- If $\tau = \tau' + 1$ is successor and$o(\alpha_{x_k}) = \tau'$ then $s$ is a maximal branch in $T$.
- If $\tau$ is limit or $\tau = \tau' + 1$ but $o(\alpha_{x_k}) < \tau'$, then $\mathrm{Succ}_T(s) \in \bigcap\{U_{\alpha,\gamma'}(x_k) \mid o(\alpha_{x_k}) < \gamma' < \tau\}$ and $x_k \prec y$ for every $y \in \mathrm{Succ}_T(s)$.

It remains to define the ultrafilters $U_{\alpha,\gamma'}(x)$ for every $\gamma-$coherent $x$ with $\gamma < \gamma' < \tau$. They are defined by induction on $\gamma'$. We note that $U_{\alpha,\gamma'}(x)$ will concentrate on the set of all $y \in \mathcal{P}_\alpha(\alpha^{+\gamma'})$ so that $\bar{x} = x^{\alpha_y}_\gamma$.
Let $j_{\alpha,\gamma'} : V \to M_{\alpha,\gamma'} \cong \mathrm{Ult}(V, U_{\alpha,\gamma'})$ be the ultrapower embedding. Working in $M_{\alpha,\gamma'}$, we use the fact that the iteration satisfies the Prikry condition (see [9]) and that the end tail of the iteration $R = j_{\alpha,\gamma'}(\mathbb{P}_\alpha) \setminus (\alpha + 1)$ is $(2^\alpha)^+-$closed to define a sequence of master conditions $\langle r_\eta \mid \eta < 2^\alpha\rangle$ deciding all the $R-$statements $j_{\alpha,\gamma'}$"$\alpha^{+\gamma'} \in j_{\alpha,\gamma'}(\dot{X})$ for every $\mathbb{P}_\alpha$ name $\dot{X}$ for a subset of $\mathcal{P}_\alpha(\alpha^{+\gamma'})$.
Define $U_{\alpha,\gamma'}(x)$ as follows. Let $X \subset \mathcal{P}_\alpha(\alpha^{+\gamma'})$ and $\dot{X}$, a $\mathbb{P}_\alpha$ name for $X$. $X \in U_{\alpha,\gamma'}(x)$ if and only if there is some $p \in G \subset \mathbb{P}_\alpha$, $\eta < 2^\alpha$ and a tree $T$ so that

$$p^\frown\langle x, T\rangle^\frown r_\eta \Vdash j_{\alpha,\gamma'}\text{“}\alpha^{+\gamma'} \in j_{\alpha,\gamma'}(\dot{X})$$

Extensions and direct extensions are defined as usual. The arguments in [9] show that $U_{\alpha,\gamma'}(x)$ is a $\alpha-$complete fine ultrafilter on $\mathcal{P}_\alpha(\alpha^{+\gamma'})$ and that the poset $\mathbb{Q}_{\alpha,\tau}$ satisfies the Prikry property.
Let $H_{\alpha,\tau} \subset \mathbb{Q}_{\alpha,\tau}$ be a generic filter. For every $\gamma < \tau$ there exists a unique $\gamma-$coherent $x$ so that $\langle x, T\rangle \in H_{\alpha,\tau}$ for some tree $T$. Denote $x$ by $x^\alpha_\gamma$. Let $\vec{x}^{\alpha,\tau} = \langle x^\alpha_\gamma \mid \gamma < \tau\rangle$ be the induced generic sequence. Since all the measures $U_{\alpha,\gamma'}(x)$ are fine, a density argument shows that if $\tau$

is limit then $\alpha^{+\tau} = \bigcup_{\gamma<\tau} x^\alpha_\gamma$.

We finally define $\mathbb{Q}_\alpha = \mathbb{Q}^\lambda_\alpha$.

(3) Suppose that $\alpha < \kappa$ and $G_{\alpha+1} \subset \mathbb{P}_{\alpha+1}$ is a generic filter. The arguments in [21] show that if $o(\alpha)$ is a limit ordinal then $\Box^*_\alpha$ fails in $V[G_{\alpha+1}]$. Since the rest of the iteration does not add new subset to $\alpha^+$, it follows there is no $\Box^*_\alpha$ sequence in a $\mathbb{P}_\kappa$ generic extension.

(4) Let $G \subset \mathbb{P}_\kappa$ be a generic filter and let $\Delta = \bigcup\{\Delta_\tau \mid \tau < \kappa \text{ limit }\}$. It follows that $\Delta$ is a fat stationary set in $V[G]$ and $\Box^*_\alpha$ fails at every $\alpha \in \Delta$. Let $\mathbb{R}$ be the forcing which adds a club to $\Delta$ by initial segments. By Abraham-Shelah [**?**] $\mathbb{R}$ is $<\kappa-$distributive and therefore does not collapse cardinals nor does it add a weak square to a cardinal $\alpha \in \Delta$. Furthermore, $\kappa$ remains Mahlo in a $\mathbb{R}$ generic extension $V[G][H]$ of $V[G]$. Let $N = (V[G][H])_\kappa = V_\kappa^{V[G*H]}$. Then $N$ is a model of GB (Godel-Bernays) which contains a club $C$ through its ordinals $\alpha$ which doe not carry $\Box^*_\alpha$ sequence.

## 6. Conclusion

In a forthcoming paper of the third author [**?**], a model is constructed where $\aleph_{\omega^2}$ is strong limit and weak square fails for all cardinals in the interval $[\aleph_1, \aleph_{\omega^2+2}]$. In particular, it is shown that one can put collapses between the Prikry points of the Gitik-Sharon [12] construction which enforce the failure of weak s quare at cardinals below $\kappa$ while making $\kappa$ into $\aleph_{\omega^2}$.

It is reasonable to believe that this construction could be combined with the forcing from the present paper, but we are left with the unsatisfactory result that weak square will hold at some successors of singulars in the extension. To make this precise we formulate a question which seems to capture the limit of a naive combination of the two techniques.

**Question 6.1.** *Suppose that $\kappa$ is a singular cardinal of cofinality $\omega$ such that $\Box^*_\lambda$ fails for all $\lambda \in [\aleph_1, \kappa)$ and $|\{\lambda < \kappa \mid \lambda \text{ is singular strong limit }\}| = \kappa$. Is there a $\Box^*_\kappa$-sequence?*

**Question 6.2.** *Are there versions of Theorems 1.1 and 1.2 where the failure of $\Box^*_\nu$ is replaced with the tree property at $\nu^+$?*

**Question 6.3.** *Let $C_G \subset \kappa$ be a generic Radin club added by the poset $\mathbb{P}$ defined in Section 3. Does $\Box_{\nu,\omega}$ fail at every ordinal $\nu \in C_G$?*

## References

[1] Uri Abraham. Aronszajn trees on $\aleph_2$ and $\aleph_3$. *Annals of Pure and Applied Logic*, 24(3):213 – 230, 1983.

[2] James Cummings. A model in which GCH holds at successors but fails at limits. *Trans. Amer. Math. Soc.*, 329(1):1–39, 1992.

[3] James Cummings and Matthew Foreman. The tree property. *Advances in Mathematics*, 133(1):1 – 32, 1998.

[4] James Cummings and Matthew Foreman. Diagonal Prikry extensions. *J. Symbolic Logic*, 75(4):1383–1402, 2010.

[5] Mirna Džamonja and Saharon Shelah. On squares, outside guessing of clubs and $I_{<f}[\lambda]$. *Fund. Math.*, 148(2):165–198, 1995.

[6] Paul Erdös and Alfred Tarski. On some problems involving inaccessible cardinals. *Essays on the Foundations of Mathematics*, pages 50–82, 1961.

[7] Matthew Foreman. More saturated ideals. In *Cabal seminar 79–81*, volume 1019 of *Lecture Notes in Math.*, pages 1–27. Springer, Berlin, 1983.

[8] Matthew Foreman and W. Hugh Woodin. The generalized continuum hypothesis can fail everywhere. *Ann. of Math. (2)*, 133(1):1–35, 1991.

[9] Moti Gitik. Changing cofinalities and the nonstationary ideal. *Israel Journal of Mathematics*, 56(3):280–314, 1986.

[10] Moti Gitik. The strength of the failure of the singular cardinal hypothesis. *Ann. Pure Appl. Logic*, 51(3):215–240, 1991.

[11] Moti Gitik. Prikry-type forcings. In *Handbook of set theory*, pages 1351–1447. Springer, 2010.

[12] Moti Gitik and Assaf Sharon. On SCH and the approachability property. *Proc. Amer. Math. Soc.*, 136(1):311–320, 2008.

[13] Menachem Magidor. On the singular cardinals problem. I. *Israel J. Math.*, 28(1-2):1–31, 1977.

[14] Menachem Magidor. Changing cofinality of cardinals. *Fund. Math.*, 99(1):61–71, 1978.

[15] Menachem Magidor and Saharon Shelah. The tree property at successors of singular cardinals. *Arch. Math. Logic*, 35(5-6):385–404, 1996.

[16] William Mitchell. Aronszajn trees and the independence of the transfer property. *Ann. Math. Logic*, 5:21–46, 1972/73.

[17] D Monk and D Scott. Additions to some results of Erdös and Tarski. *Fundamenta Mathematicae*, 53(3):335–343, 1964.

[18] Itay Neeman. Aronszajn trees and failure of the singular cardinal hypothesis. *J. Math. Log.*, 9(1):139–157, 2009.

[19] Itay Neeman. The tree property up to $\aleph_{\omega+1}$. *J. Symb. Log.*, 79(2):429–459, 2014.

[20] Lon Berk Radin. Adding closed cofinal sequences to large cardinals. *Annals of Mathematical Logic*, 22(3):243 – 261, 1982.

[21] Dima Sinapova. A model for a very good scale and a bad scale. *The Journal of Symbolic Logic*, 73(04):1361–1372, 2008.

[22] Dima Sinapova. The tree property and the failure of SCH at uncountable cofinality. *Arch. Math. Logic*, 51(5-6):553–562, 2012.

[23] Dima Sinapova. The tree property and the failure of the singular cardinal hypothesis at $\aleph_{\omega^2}$. *J. Symbolic Logic*, 77(3):934–946, 2012.

[24] Dima Sinapova and Spencer Unger. Scales at $N_\omega$. *Israel J. Math.*, 209(1):463–486, 2015.

[25] E. Specker. Sur un problème de Sikorski. *Colloquium Mathematicum*, 2:9–12, 1949.

[26] Spencer Unger. The tree property below $\aleph_{\omega\cdot 2}$. *Annals of Pure and Applied Logic*, 167(3):247 – 261, 2016.