WEAK PREDICTION PRINCIPLES

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Abstract. We prove the consistency of the failure of the weak diamond Φ_λ at strongly inaccessible cardinals. On the other hand we show that the very weak diamond Ψ_λ is equivalent to the statement $2^{<\lambda} < 2^\lambda$ and hence holds at every strongly inaccessible cardinal.

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0. Introduction

The prediction principle $\Diamond_\lambda$ (diamond on λ) was discovered by Jensen, [5], who proved that it holds over any regular cardinal λ in the constructible universe. This principle says that there exists a sequence $\langle A_\alpha : \alpha < \lambda \rangle$ of sets, $A_\alpha \subseteq \alpha$ for every $\alpha < \lambda$, such that for every $A \subseteq \lambda$ the set $\{\alpha < \lambda : A \cap \alpha = A_\alpha\}$ is stationary.

In the ancient era, the main focus was the case of $\lambda = \aleph_1$. It is immediate that $\Diamond_{\aleph_1} \Rightarrow 2^{\aleph_0} = \aleph_1$, but consistent that $2^{\aleph_0} = \aleph_1$ along with $\neg\Diamond_{\aleph_1}$. Motivated by algebraic constructions, Devlin and Shelah [2] introduced a weak form of the diamond principle which follows from the continuum hypothesis:

Definition 0.1. The Devlin-Shelah weak diamond.
Let λ be a regular uncountable cardinal.
The weak diamond on λ (denoted by Φ_λ) is the following principle:
For every function $c : {}^{<\lambda}2 \to 2$ there exists a function $g \in {}^{\lambda}2$ such that $\{\alpha \in \lambda : c(f \restriction \alpha) = g(\alpha)\}$ is a stationary subset of λ whenever $f \in {}^{\lambda}2$.

The idea is that we replace the prediction of the initial segments of a set (or a function) by predicting only their color. The function c is a coloring, and the function g is the weak diamond function which gives stationarily many guesses for the c-color of the initial segments of every function f. It is easy to see that the real diamond implies the weak diamond.

Concerning cardinal arithmetic, not only that $\Phi_{\aleph_1}$ follows from the continuum hypothesis, but actually $2^{\aleph_0} < 2^{\aleph_1}$ implies $\Phi_{\aleph_1}$ as proved in [2]. On the other hand, $\Phi_{\aleph_1}$ implies $2^{\aleph_0} < 2^{\aleph_1}$ (as noted by Uri Abraham) so both assertions are equivalent.

The diamond and the weak diamond are prediction principles, while $2^{\aleph_0} < 2^{\aleph_1}$ and $2^{\aleph_0} = \aleph_1$ belong to cardinal arithmetic. The above statements show that a simple connection may exist. Actually, if $\lambda > \aleph_1$ and $\lambda = \kappa^+$ then the situation becomes even simpler. For the diamond, $2^\kappa = \kappa^+ \Leftrightarrow \Diamond_{\kappa^+}$ (see [8]), and for the weak diamond $2^\kappa < 2^{\kappa^+} \Leftrightarrow \Phi_{\kappa^+}$ (a straightforward generalization of [2]). The negation of $\Diamond_{\aleph_1}$ with the affirmation of $2^{\aleph_0} = \aleph_1$ has been recognized as a peculiarity of the first uncountable cardinal.

However, the situation is totally different if $\lambda = \mathrm{cf}(\lambda)$ is a limit cardinal. One direction is still easy and has a general nature. If $\Diamond_\lambda$ then $2^{<\lambda} = \lambda$ and if Φ_λ then $2^{<\lambda} < 2^\lambda$ (see Claim 1.1 below). Notice that these formulations coincide with the above description of the successor case $\lambda = \kappa^+$. But can we prove the opposite implication?

If $\lambda = \mathrm{cf}(\lambda)$ is a limit cardinal then λ is a large cardinal (in the philosophical sense; its existence cannot be established in ZFC). Now if λ is large enough then $\Diamond_\lambda$ holds. Measurability suffices, and actually much less. If λ is ineffable (a property which may live happily with V = L) or even subtle then $\Diamond_\lambda$ and hence also Φ_λ.

A striking (and unpublished) result of Woodin shows that small large cardinals manifest much more interesting demeanor. Woodin proved the consistency of a strongly inaccessible cardinal λ for which $\diamondsuit_\lambda$ fails. Moreover, the construction can be strengthened to strongly Mahlo. In the current paper we prove the same assertion, upon replacing the diamond by the weak diamond. For both the diamond and the weak diamond we do not know what happens if λ is weakly compact.

We conclude that the cardinal arithmetic assumption $2^{<\lambda} < 2^\lambda$ is strictly weaker than the prediction principle Φ_λ. It is still tempting to look for a prediction principle which is characterized by $2^{<\lambda} < 2^\lambda$. We define the following:

Definition 0.2. The very weak diamond.
Let λ be an uncountable cardinal.
The very weak diamond on λ (denoted by Ψ_λ) is the following principle:
For every function $c : {}^{<\lambda}2 \to 2$ there exists a function $g \in {}^{\lambda}2$ such that $\{\alpha \in \lambda : c(f \upharpoonright \alpha) = g(\alpha)\}$ is an unbounded subset of λ whenever $f \in {}^{\lambda}2$.

As we shall see, $\Psi_\lambda \Leftrightarrow 2^{<\lambda} < 2^\lambda$ whenever $\lambda = \mathrm{cf}(\lambda) > \aleph_0$. Let us mention Galvin's property which follows from $2^{<\lambda} < 2^\lambda$ but consistent with $2^{<\lambda} = 2^\lambda$ (see [3]). Clumping all these principles together we have a systematic hierarchy of weak prediction principles, each of which implied by the stronger one but strictly weaker from the next stage.

The basic tool for proving $\neg\Phi_\lambda$ over small large cardinals is Radin forcing, [6]. Since there are several ways to introduce this forcing notion we indicate that our approach is taken from [4], and in particular we use the Jerusalem notation i.e. $p \leq_{\mathbb{R}} q$ means that q is stronger than p.

1. Weak diamond and Radin forcing

We commence with the easy direction about the connection between Φ_λ and cardinal arithmetic. Actually, the claim below applies to Ψ_λ as well. A parallel assertion can be proved easily for $\diamondsuit_\lambda$.

Claim 1.1. *The basic claim.*
Let λ be an uncountable cardinal.
If Φ_λ (or even Ψ_λ) holds then $2^{<\lambda} < 2^\lambda$.

Proof.
Assume towards contradiction that Φ_λ holds and $2^{<\lambda} = 2^\lambda$. Let $b : 2^{<\lambda} \to 2^\lambda$ be a surjection, such that for every $\alpha < \lambda$ and every $t \in {}^\alpha 2$, if there is an ordinal $\delta < \alpha$ such that $\varepsilon \in [\delta, \alpha) \Rightarrow t(\varepsilon) = 0$ then $b(t) = b(t \upharpoonright \delta)$. We are trying to describe a coloring $F : {}^{<\lambda}2 \to 2$, which exemplifies the failure of the weak diamond.

Assume $\alpha < \lambda$ and $\eta \in {}^\alpha 2$. Set $F(\eta) = [b(\eta)](\alpha)$. By Φ_λ we can choose a function $g \in {}^\lambda 2$ which predicts F. Let $h \in {}^\lambda 2$ be the opposite function, i.e. $h(\alpha) = 1 - g(\alpha)$ for every $\alpha < \lambda$. Let $t \in {}^{<\lambda}2$ be any mapping for which $b(t) = h$. Let $\delta < \lambda$ be such that $t \in {}^\delta 2$. We define $f \in {}^\lambda 2$ as an extension of t as follows. If $\alpha < \delta$ then $f(\alpha) = t(\alpha)$ and if $\alpha \geq \delta$ then $f(\alpha) = 0$. Observe that $b(f \upharpoonright \alpha) = b(f \upharpoonright \delta)$ for every $\alpha \in (\delta, \lambda)$. Consequently, $F(f \upharpoonright \alpha) = [b(f \upharpoonright \alpha)](\alpha) = [b(f \upharpoonright \delta)](\alpha) = [b(t)](\alpha) = h(\alpha) \neq g(\alpha)$, so g fails to predict $F(f \upharpoonright \alpha)$ on an end-segment of λ, a contradiction.

Notice that the above argument shows that even the very weak diamond Ψ_λ implies $2^{<\lambda} < 2^\lambda$, so the proof is accomplished.

$\square_{1.1}$

It follows from the above claim that the weak diamond and the very weak diamond are equivalent in the successor case. We comment that the same holds for the common diamond over successor cardinals, if one replaces the requirement of stationary set of guesses by an unbounded set.

Corollary 1.2. *Φ_λ and Ψ_λ.*
If $\lambda = \kappa^+$ then $\Phi_\lambda \Leftrightarrow \Psi_\lambda$. The same holds true if λ is weakly inaccessible and $2^{<\lambda} = 2^\kappa$ for some $\kappa < \lambda$.

Proof.
The implication $\Phi_\lambda \Rightarrow \Psi_\lambda$ results from the definition. For the opposite dierction, if Ψ_λ then $2^{<\lambda} < 2^\lambda$ by the above claim, and hence Φ_λ by [2].

$\square_{1.2}$

Claim 1.1 gives one direction. As we shall see in Theorem 1.6 below, the other direction cannot be proved. Indeed, every strongly inaccessible cardinal satisfies $2^{<\lambda} < 2^\lambda$, but the negation of Φ_λ can be forced over some strongly inaccessible cardinal. However, we can prove that $2^{<\lambda} < 2^\lambda$ is equivalent to the very weak diamond Ψ_λ. We shall use the fact that if λ is weakly inaccessible and $2^{<\lambda} = 2^\kappa < 2^\lambda$ for some $\kappa < \lambda$ then Φ_λ. This fact appears already in [2], without explicit proof. Since it plays a key-role in the theorem below, we spell out the proof:

Claim 1.3. *Weak diamond out of nowhere.*
If λ is weakly inaccessible, $\lambda < 2^{<\lambda} < 2^\lambda$ and $2^\kappa = 2^{<\lambda}$ for some $\kappa < \lambda$, then Φ_λ.

Proof.
Assume towards contradiction that $\neg(\Phi_\lambda)$, and choose a coloring $F : {}^{<\lambda}2 \to 2$ which exemplifies it. For every $g \in {}^\lambda 2$ we can find $f \in {}^\lambda 2$ so that $\{\alpha < \lambda : F(f \restriction \alpha) = g(\alpha)\} \in \mathcal{D}_\lambda$. Indeed, given any $g \in {}^\lambda 2$ we choose $f \in {}^\lambda 2$ for which $N = \{\alpha < \lambda : F(f \restriction \alpha) = 1 - g(\alpha)\}$ is not a stationary subset of λ. This can be done by the assumption $\neg(\Phi_\lambda)$ with respect to the function $1 - g$. However, in this case $\lambda \setminus N \in \mathcal{D}_\lambda$ and for every $\alpha \in \lambda \setminus N$ we have $F(f \restriction \alpha) \neq 1 - g(\alpha)$ and hence $F(f \restriction \alpha) = g(\alpha)$.

Let S be the collection of all the sequences of the form $(\alpha, \ldots, g_\nu, f_\nu, \ldots)_{\nu<\beta}$ when $\alpha, \beta < \lambda$ and $g_\nu, f_\nu \in {}^\alpha 2$ for every $\nu < \beta$. Observe that $|S| = 2^{<\lambda}$, which equals 2^κ by the assumption of the theorem. Let h be a one-to-one mapping between S and ${}^\kappa 2$.

Assume $g \in {}^\lambda 2$ is any function. We describe a process of defining $\kappa \cdot \kappa$-many functions $g_\nu, f_\nu \in {}^\lambda 2$ such that every pair of functions is a pair of many guesses. We also choose a club subset of λ which is contained in the set of these guesses. This is done by induction on $\eta < \kappa$, when we choose in the η-th stage $\kappa \cdot \eta$ many functions and one additional pair of functions at $\eta = 0$.

For $\eta = 0$ let $g_0 = g, f_0 = f$ for some f which is guessed by g on a club C. We set $C_0 = C$. If η is a limit ordinal then we let $C_\eta = \bigcap\limits_{\zeta<\eta} C_\zeta$. For the functions, we just take all the functions that were defined in previous stages. The interesting case is $\eta + 1$, assuming that the stage of η has been accomplished.

For each $\alpha < \lambda$ let $\beta_{\alpha,\eta}$ be the first member of C_η greater than α. We define simultaneously the sequence of functions $\langle g_{\kappa\cdot\eta+\delta} : \delta < \kappa\rangle$. For this end, we have to determine the value of all these functions for every $\alpha < \lambda$, so $\langle g_{\kappa\cdot\eta+\delta}(\alpha) : \delta < \kappa\rangle = h(\beta_{\alpha,\eta}, \ldots, g_\nu \restriction \beta_{\alpha,\eta}, f_\nu \restriction \beta_{\alpha,\eta}, \ldots)_{\nu<\kappa\cdot\eta}$.

Having the functions $g_{\kappa\cdot\eta+\delta}$ at hand for every $\delta < \kappa$, we choose for each one of them a function $f_{\kappa\cdot\eta+\delta} \in {}^\lambda 2$ such that $A_{\kappa\cdot\eta+\delta} = \{\alpha < \lambda : F(f_{\kappa\cdot\eta+\delta} \restriction \alpha) = g_{\kappa\cdot\eta+\delta}(\alpha)\} \in \mathcal{D}_\lambda$. Finally, we choose a club $C_{\eta+1}$ of λ such that $C_{\eta+1} \subseteq \bigcap\limits_{\delta<\kappa} A_{\kappa\cdot\eta+\delta} \cap C_\eta$. The above process can be rendered for every function $g \in {}^\lambda 2$. Let $\langle g_\nu : \nu < \kappa \cdot \kappa\rangle$ enumerate all the functions derived from g. We shall add g as a superscript to all the functions and club subsets which emerge out of g, e.g. g^g_ν and so forth.

We define an equivalence relation E on the members of ${}^\lambda 2$ by gEg' iff:

(*a*) $\gamma = \min(\bigcap\{C^g_\eta : \eta < \kappa\}) = \min(\bigcap\{C^{g'}_\eta : \eta < \kappa\})$.
(*b*) $g^g_\nu \restriction \gamma = g^{g'}_\nu \restriction \gamma, f^g_\nu \restriction \gamma = f^{g'}_\nu \restriction \gamma$ for every $\nu < \kappa \cdot \kappa$.

The number of equivalence classes of E is bounded by $\lambda \cdot 2^{<\lambda} = 2^{<\lambda} < 2^\lambda$. Consequently, we can choose two functions $g, g' \in {}^\lambda 2$ such that $g \neq g'$ and

gEg'. For the rest of the proof we focus on these two functions, hence we write C_η instead of C^g_η and C'_η instead of $C^{g'}_\eta$. The same convention is applied to the derived functions. Let C be $\bigcap\{C_\eta : \eta < \kappa\}$ and let $C' = \bigcap\{C'_\eta : \eta < \kappa\}$. Enumerate the members of C in an increasing order by $\{\gamma_\xi : \xi < \lambda\}$. Similarly, $\{\gamma'_\xi : \xi < \lambda\}$ would be the increasing enumeration of C'. Notice that $\gamma_0 = \gamma'_0 = \gamma$.

We try to prove by induction on $\xi < \lambda$ that $\gamma_\xi = \gamma'_\xi$ (hence $C = C'$) and that all the functions coincide (i.e., requirement (b) holds for every γ_ξ). If we succeed, the proof will be accomplished. Indeed, considering $\nu = 0$ we shall have $g \restriction \gamma_\xi = g_0 \restriction \gamma_\xi = g'_0 \restriction \gamma_\xi = g' \restriction \gamma_\xi$ for every $\xi < \lambda$ and hence $g = g'$, a contradiction.

The case $\xi = 0$ is actually the content of the definition of E, as $\gamma_0 = \gamma$. Part (b) of the definition of E is responsible to the equality of the functions restricted to γ, so we can start the induction. The case of a limit ordinal ξ is also simple. By the induction hypothesis, $\gamma_\zeta = \gamma'_\zeta$ for every $\zeta < \xi$. Now $\gamma_\xi = \bigcup\limits_{\zeta<\xi} \gamma_\zeta = \bigcup\limits_{\zeta<\xi} \gamma'_\zeta = \gamma'_\xi$ (recall that both C, C' are closed). For the functions, if $\nu < \lambda$ then $g_\nu \restriction \gamma_\xi = \bigcup\limits_{\zeta<\xi} g_\nu \restriction \gamma_\zeta$, and the same holds for g'_ν, so the induction hypothesis applies. A similar argument shows that the f_ν-s coincide.

The remaining case is $\xi + 1$. For every $\alpha < \lambda$ and every $\eta < \kappa$ let $s_{\alpha,\eta}$ be the sequence $(\beta_{\alpha,\eta}, \ldots, g_\nu \restriction \beta_{\alpha,\eta}, f_\nu \restriction \beta_{\alpha,\eta}, \ldots)_{\nu<\kappa\cdot\eta}$. We define $s'_{\alpha,\eta}$ in the same manner, with respect to g'_ν, f'_ν. By the induction hypothesis:

$$\begin{gathered} h(s_{\gamma_\xi,\eta}) = \langle g_{\kappa\cdot\eta+\delta}(\gamma_\xi) : \delta < \kappa\rangle = \\ \langle F(f_{\kappa\cdot\eta+\delta} \restriction \gamma_\xi) : \delta < \kappa\rangle = \langle F(f'_{\kappa\cdot\eta+\delta} \restriction \gamma_\xi) : \delta < \kappa\rangle \\ = \langle g'_{\kappa\cdot\eta+\delta}(\gamma_\xi) : \delta < \kappa\rangle = h(s'_{\gamma_\xi,\eta}). \end{gathered}$$

However, the function h is one-to-one, and hence $s_{\gamma_\xi,\eta} = s'_{\gamma_\xi,\eta}$. In particular, $\beta_{\gamma_\xi,\eta} = \beta'_{\gamma_\xi,\eta}$ for every $\eta < \kappa$ and hence $\gamma_{\xi+1} = \gamma'_{\xi+1}$. The equality between $s_{\gamma_\xi,\eta}$ and $s'_{\gamma_\xi,\eta}$ says that the derived functions coincide as well, so we are done.

$\square_{1.3}$

Remark 1.4. Another way to phrase the idea in the above proof is by noticing that if $2^{<\lambda} = 2^\kappa$ then $\neg\Phi_\lambda$ codes a one-to-one mapping from 2^λ into $2^{<\lambda}$, and hence $2^{<\lambda} = 2^\lambda$.

$\square_{1.4}$

Now we can prove the following:

Theorem 1.5. *Very weak diamond and cardinal arithmetic.*
For every regular uncountable cardinal λ we have $2^{<\lambda} < 2^\lambda$ iff Ψ_λ.

Proof.
By Corollary 1.2 we may assume that λ is a limit cardinal. If Ψ_λ holds

then $2^{<\lambda} < 2^\lambda$ holds by Claim 1.1. For the opposite direction, assume that $2^{<\lambda} < 2^\lambda$. If $2^{<\lambda} = 2^\kappa = 2^\lambda$ for some $\kappa < \lambda$ then Φ_λ holds by Claim 1.3 and hence also Ψ_λ. So assume that this is not the situation (as always happens in the case of a strongly inaccessible cardinal).

We claim that Φ_α holds for unbounded set of α-s below λ. For proving this assertion, let us define an increasing continuous sequence of cardinals $\langle \alpha_\varepsilon : \varepsilon < \lambda \rangle$ as follows. For $\varepsilon = 0$ let θ be the first cardinal for which $2^{\aleph_0} < 2^\theta$. Such a cardinal must be below λ by our assumption, and we let $\alpha_0 = \theta$. For a limit ordinal ε let α_ε be $\bigcup\limits_{\zeta<\varepsilon} \alpha_\zeta$. Finally, if $\varepsilon = \zeta + 1$ let $\theta < \lambda$ be the first cardinal for which $2^{\aleph_\zeta} < 2^\theta$. Define $\alpha_\varepsilon = \theta$.

We claim that $\Phi_{\alpha_\varepsilon}$ holds whenever ε is a successor ordinal (and actually also when $\varepsilon = 0$). Indeed, if $\varepsilon = \zeta + 1$ and $\alpha_\varepsilon = \gamma^+$ for some $\gamma < \lambda$ then $2^\gamma < 2^{\gamma^+} = 2^{\alpha_\varepsilon}$ and then $\Phi_{\alpha_\varepsilon}$ holds as indicated in the introduction. If α_ε is a limit cardinal then it cannot be singular (by the Bukovsky-Hechler theorem). It cannot be strongly inaccessible (since it has been chosen as the first) and hence it must be weakly inaccessible. Notice, however, that it satisfies the assumptions of Claim 1.3, since the sequence $\langle 2^\gamma : \gamma < \alpha_\varepsilon \rangle$ is constant from some point below α_ε onwards. By claim 1.3 we have $\Phi_{\alpha_\varepsilon}$.

We claim now that Ψ_λ holds. For this, let $c : {}^{<\lambda} \rightarrow 2$ be a coloring. For every $\zeta < \lambda$ let $\varepsilon = \zeta + 1$ and let c_ε be the restriction $c \restriction \alpha_{\zeta+1}$. Choose a function g_ε which exemplifies $\Phi_{\alpha_\varepsilon}$ with respcet to c_ε for every $\varepsilon = \zeta + 1 < \lambda$. Define $h : \lambda \rightarrow \lambda$ as follows. For every $\beta < \lambda$ let $h(\beta)$ be the first ordinal $\varepsilon < \lambda$ so that $\alpha_\varepsilon \leq \beta < \alpha_{\varepsilon+1}$.

We define $g \in {}^\lambda 2$ as follows. Given $\beta < \lambda$, if $h(\beta) = \varepsilon$ is a limit ordinal then $g(\beta) = 0$. If $h(\beta) = \varepsilon$ is a sucessor ordinal then $g(\beta) = g_\varepsilon(\beta)$. Let us show that g exemplifies Ψ_λ.

Assume $f \in {}^\lambda 2$. For every successor oridnal $\varepsilon = \zeta + 1 < \lambda$ let $f_\varepsilon = f \restriction \alpha_\varepsilon$. By $\Phi_{\alpha_\varepsilon}$ we can choose an ordinal $\beta_\varepsilon \in [\alpha_\zeta, \alpha_\varepsilon)$ for which $g_\varepsilon(\beta_\varepsilon) = c_\varepsilon(f_\varepsilon \restriction \beta_\varepsilon)$. By the above definitions it folows that $g(\beta_\varepsilon) = c(f \restriction \beta_\varepsilon)$. Since we have unboundedly many β_ε of this form, we are done.

$\square_{1.5}$

Our next goal is to demonstrate the fact that Ψ_λ is strictly weaker than Φ_λ. We shall use Radin forcing $R(\vec{U})$, when $\vec{U} = \langle \kappa \rangle ^\frown \langle U_\tau \mid \tau < \kappa^+ \rangle$ is a measure sequence for some measurable cardinal κ with $o(\kappa) \geq \kappa^+$. Recall that conditions in $R(\vec{U})$ are finite sequences $p = \langle d_i \mid i \leq k \rangle$ satisfying the following conditions.

(א) $\vec{d} = \langle d_i \mid i < k \rangle$ is a finite sequence. For every $i \leq k$, d_i is either of the form $\langle \kappa_i \rangle$ where $\kappa_i < \kappa$ is an ordinal, or of the form $d_i = \langle \vec{\mu_i}, a_i \rangle$ where $\vec{\mu_i}$ is a measure sequence on a measurable cardinal $\kappa_i = \kappa(\vec{\mu_i}) \leq \kappa$ and $a_i \in \cap \vec{\mu_i}$.

(ב) $\langle \kappa_i \mid i < k \rangle$ is increasing.

(ג) $d_k = \langle \vec{U}, A \rangle$.

For each $i \leq k$ we denote κ_i by $\kappa(d_i)$ and a_i by $a(d_i)$. Given a condition $p = \langle d_i \mid i \leq k\rangle$ as aobve, we will frequently separate $\langle \vec{U}, A\rangle$ from the other components and write $p = \vec{d}^\frown\langle \vec{U}, A\rangle$ where $\vec{d} = \langle d_i \mid i < k\rangle$.

A condition $p^* = \langle d^*_i \mid i \leq k^*\rangle$ is a direct extension of $p = \langle d_i \mid i \leq k\rangle$ if $k^* = k$ and $a(d^*_i) \subset a(d_i)$ whenever $a(d_i)$ exists. A condition p' is a one point extension of p if there exists $j \leq k$ and a measure sequence $\vec{\nu} \in a(d_j)$ with $\kappa(\vec{\nu}) > \kappa(d_{j-1})$ and $p' = p^\frown\langle\vec{\nu}\rangle$ is either $\langle d_i \mid i < j\rangle^\frown\langle\vec{\nu}\rangle^\frown\langle d_i \mid i \geq j\rangle$ if $\vec{\nu} = \alpha$ is an ordinal, or $\langle d_i \mid i < j\rangle^\frown\langle\vec{\nu}, a(d_j) \cap V_{\kappa(\vec{\nu})}\rangle\rangle^\frown\langle d_i \mid i \geq j\rangle$ if $\vec{\nu}$ is a nontrivial measure sequence. A condition q extends p if it is obtained from p by a finite sequence of one point extensions and direct extensions.

Let $\vec{U}$ be a measure sequence. It is shown in [4] that $R(\vec{U})$ is a Prikry type forcing notion, preserves all cardinals and satisfies $\kappa(\vec{U})^+ - cc$. It is also proved that κ remains strongly inaccessible in a generic extension by $R(\vec{U})$. Suppose $G \subset R(\vec{U})$ is a generic filter. Let $MS_G \subset MS$ be the set of all $\vec{u} \in MS$ for which there exists some $p \in G$ of the form $p = \vec{d}^\frown\langle \vec{U}, A\rangle$ such that $\vec{d} = \langle u_0, a_0\rangle, \ldots, \langle u_{k-1}, a_{k-1}\rangle$ and $\vec{u} = u_i$ for some $i < k$. Define $C_G = \{\kappa(\vec{u}) \mid \vec{u} \in MS_G\}$. C_G is the Radin club associated with G.

Theorem 1.6. *Strong inaccessibility and $\neg\Phi_\kappa$.*
Assuming the existence of a measurable cardinal κ such that $o(\kappa) \geq \kappa^+$ and $2^\kappa = 2^{\kappa^+}$, it is consistent that there is an inaccessible cardinal κ such that $\neg\Phi_\kappa$.

Proof.
Suppose that $\vec{U} = \langle\kappa\rangle^\frown\langle U_\tau \mid \tau < \kappa^+\rangle$ is a measure sequence of a measurable cardinal κ derived from an elementary embedding $j : V \to M$ satisfying $M \models 2^\kappa = 2^{\kappa^+}$. Therefore, $U_\tau = \{X \subset V_\kappa \mid \vec{U} \restriction \tau \in j(X)\}$ for all $\tau < \kappa^+$. Clearly, $M \models 2^\kappa = 2^{\kappa^+} = |([V_\kappa]^{<\omega} \times \mathcal{P}(V_\kappa))^{\kappa\times\kappa^+}|$. In V, let $H : \kappa \to V_\kappa$ be a partial function, $\mathrm{dom}(H) = \{\alpha < \kappa \mid 2^\alpha = 2^{\alpha^+}\}$. For every $\alpha \in \mathrm{dom}(H)$, $H_\alpha : 2^\alpha \leftrightarrow ([V_\alpha]^{<\omega} \times \mathcal{P}(V_\alpha))^{\alpha\times\alpha^+}$ is a bijection. Let $H_\kappa = j(H)(\kappa) : 2^\kappa \leftrightarrow ([V_\kappa]^{<\omega} \times \mathcal{P}(V_\kappa))^{\kappa\times\kappa^+}$.

Let $R(\vec{U})$ be the Radin forcing associated with $\vec{U}$. Choose a generic set $G \subseteq R(\vec{U})$. We claim there is no weak diamond on κ in $V[G]$. To show this, it will be convenient to identify conditions $p = \vec{d}^\frown\langle\vec{U}, A\rangle \in R(\vec{U})$ with pairs $\langle\vec{d}, A\rangle \in V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa)$. We say that $\langle\vec{d}, A\rangle \in V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa)$ is a simple representation of p. Since $R(\vec{U})$ satisfies $\kappa^+.c.c$ we can represent antichians in $R(\vec{U})$ using elements in $(V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa))^\kappa$. We use the phrase simple representation in this context as well.

We first work in V. Let $\underset{\sim}{g}$ be an $R(\vec{U})$ name for a function from κ to 2 and $p = \vec{d}^\frown\langle\vec{U}, A\rangle$ a condition. For every $\vec{\mu} \in A$, consider the one point extension $p^\frown\langle\vec{\mu}\rangle = \vec{d}^\frown\langle\vec{\mu}, A \cap V_{\kappa(\vec{\mu})}\rangle^\frown\langle\vec{U}, A\rangle$. The forcing $R(\vec{U})/(p^\frown\langle\vec{\mu}\rangle)$ factors into a product $R(\vec{\mu}) \times R(\vec{U})/(\langle\vec{U}, A \setminus V_{\kappa(\vec{\mu})} + 1\rangle)$. It follows that $p^\frown\vec{\mu}$ has a direct extension of the form $\vec{d}^\frown\langle\vec{\nu}, A \cap V_{\kappa(\vec{\nu})}\rangle^\frown\langle\vec{U}, A_{\vec{\mu}}\rangle$ forcing

$\underset{\sim}{g}(\check{\kappa}(\vec{\mu})) = \sigma(\vec{\mu})$, where $\sigma(\vec{\mu})$ is a $R(\vec{\mu})$ name for an ordinal in $\{0,1\}$. Let $A^* = \Delta_{\vec{\mu}\in A} A_{\vec{\mu}} = \{\vec{\nu} \in V_\kappa \mid \vec{\nu} \in A_{\vec{\mu}} \text{ if } \kappa(\vec{\mu}) < \kappa(\vec{\nu})\}$ and $p^* = \vec{d}^\frown\langle \vec{U}, A^*\rangle$. $p^* \geq^* p$ and $p^\frown\langle\vec{\mu}\rangle \Vdash \underset{\sim}{g}(\check{\kappa}(\vec{\mu})) = \sigma(\vec{\mu})$ for all $\vec{\mu} \in A^*$.

In M, we define a function $h : \kappa^+ \to (V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa))^\kappa$. For every $\tau < \kappa^+$, let $h(\tau) \in (V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa))^\kappa$ be a simple representation of a maximal antichain $A_\tau \subset R(\vec{U} \restriction \tau)$ of conditions $q \in R(\vec{U} \restriction \tau)$ which force $j(\sigma)(\vec{U} \restriction \tau) = \check{0}$. We can identify h with an element in $(V_\kappa^{<\omega} \times \mathcal{P}(V_\kappa))^{\kappa\times\kappa^+}$. Let $f = H_\kappa^{-1}(h) : \kappa \to 2$. Back to V, define a function $F' : 2^{<\kappa} \to V_\kappa$ as follows. $F'(w)$ is a function whose domain is the collection of all measure sequences $\vec{\nu}$ with $\kappa(\vec{\nu}) = \alpha$. Note that for every $\alpha < \kappa$ and $w \in 2^\alpha$, $H_\alpha(w) \in (V_\alpha^{<\omega} \times \mathcal{P}(V_\alpha))^{\alpha\times\alpha^+}$.

We set $F'(w)(\vec{\nu}) = \{q \in R(\vec{\nu}) \mid q$ is simply represented by an element of $H_\alpha(w)(\text{len}(\vec{\nu}))\}$, where $\text{len}(\vec{\nu})$ is the length of the sequence. By our choice of $f = H_\kappa^{-1}(h)$ we see that $B = \{\vec{\nu} \in V_\kappa \mid F'(f \restriction \kappa(\vec{\nu}))(\vec{\nu})$ is a maximal antichain of $R(\vec{\nu})$ of conditions $q \Vdash \sigma(\vec{\nu}) = \check{0}\} \in \bigcap \vec{U}$.

Finally, we define $F : 2^{<\kappa} \to 2$ in $V[G]$. For every $\vec{\nu} \in MS_G$, let $G(\vec{\nu})$ denote the $R(\vec{\nu})$ generic filter induced by G. For each $w \in 2^{\kappa(\vec{\nu})}$ we set

$$F(w) = \begin{cases} 0 & \text{if } F'(w)(\vec{\nu}) \cap G(\vec{\nu}) \neq \emptyset \\ 1 & \text{otherwise.} \end{cases}$$

Define $F(w) = 0$ for every other $w \in 2^\alpha$.

We may assume G contains a condition $p^* = \vec{d}^\frown\langle\vec{U}, A^*\rangle$ as above. Let $X = \{\kappa(\vec{\nu}) \mid \vec{\nu} \in A^* \cap B\}$. $C_G \subset^* X$ since $X \in \bigcap\vec{U}$. The claim follows as $F(f \restriction \alpha) = g(\alpha)$ for every $\alpha \in X \cap C_G$.

$\square_{1.6}$

To round out the picture, we are left with the case of weakly but not strongly inaccessible cardinal. We distinguish three cases. If $2^{<\lambda} = 2^\lambda$ then we already know that $\neg\Phi_\lambda$ and if $2^{<\lambda} = 2^\kappa < 2^\lambda$ for some $\kappa < \lambda$ then Φ_λ. The remaining case is when the sequence $\langle 2^\theta : \theta < \lambda\rangle$ is not eventually constant. In which case $2^{<\lambda} < 2^\lambda$, and we do not know if the weak diamond holds (see [7], Question 1.28) though we have seen that the very weak diamond holds. It is possible to force Φ_λ in such cases:

Claim 1.7. *It is consistent that λ is weakly inaccessible, $\langle 2^\theta : \theta < \lambda\rangle$ is not eventually constant, λ is not strongly inaccessible and Φ_λ holds.*

Proof.
We begin with a strong limit cardinal λ in the ground model, aiming to blow up 2^θ for every regular uncountable $\theta < \lambda$. We Shall add Cohen subsets to any such θ, and also to λ itself. So let $\mathbb{P}$ be the Easton support product of $\text{Add}(\theta, \lambda^{+\theta+1})$ for every regular uncountable $\theta \leq \lambda$. Notice that $\mathbb{P}$ neither collapses cardinals, nor changes cofinalities.

The forcing $\mathbb{P}$ is λ^+-cc and $\aleph_1$-complete. It follows that $\langle 2^\theta : \theta < \lambda\rangle = \langle \lambda^{+\theta+1} : \theta < \lambda\rangle$ in the generic extension, so this sequence is not eventually constant. Although λ is not strongly inaccessible any more, it is still weakly inaccessible. It remains to show that Φ_λ holds in $V^{\mathbb{P}}$.

Let $\underset{\sim}{c}$ be a name of a coloring function from ${}^{<\lambda}2$ into 2. Observe that $2^{<\lambda} < 2^{\lambda} = \lambda^{+\lambda+1}$ in $V^{\mathbb{P}}$. Hence, by the chain condition, there exists a bounded subset B of $\lambda^{+\lambda+1}$ so that $\mathbb{P} = \mathbb{P} \upharpoonright B \times \mathbb{P} \upharpoonright (\lambda^{+\lambda+1} \setminus B)$ and $\underset{\sim}{c} \in V^{\mathbb{P} \upharpoonright B}$. Denote the lower part $\mathbb{P} \upharpoonright B$ by $\mathbb{Q}$ and the upper part $\mathbb{P} \upharpoonright (\lambda^{+\lambda+1} \setminus B)$ by $\mathbb{R}$. We may assume that $\mathbb{R}$ is $\aleph_1$-complete and λ-distributive. Let g be a new λ-Cohen set, added by $\mathbb{R}$.

Working in $V^{\mathbb{Q}}$, Let $\underset{\sim}{f}$ be a name of a function from λ into 2, and let $\underset{\sim}{D}$ be a name of a club subset of λ. For every condition $r \in \mathbb{R}$ we can define by induction on ω a sequence $\langle r_n : n \in \omega \rangle$ of conditions in $\mathbb{R}$ such that:

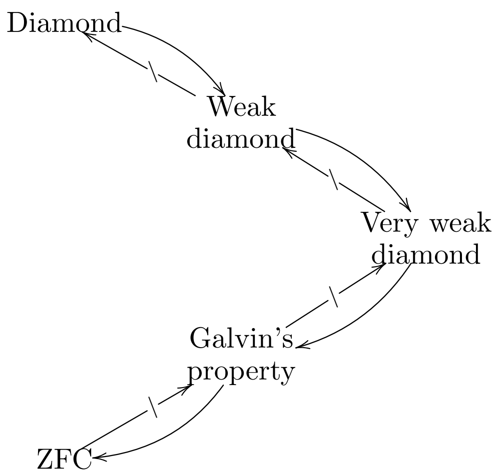
- (a) $r_0 = r$ and $r_n \leq r_{n+1}$.
- (b) $r_{n+1} \Vdash \underset{\sim}{f} \upharpoonright \alpha_n = \check{g}_n$ for some $g_n \in V^{\mathbb{Q}}$.
- (c) $\mathrm{dom}(r_{n+1}) = \alpha_{n+1}$.
- (d) $r_{n+1} \Vdash \exists \check{\beta}_n, \check{\beta}_n \in \underset{\sim}{D}$ and $\check{\alpha}_n < \check{\beta}_n < \check{\alpha}_{n+1}$.

Let p be $\bigcup_{n\in\omega} r_n$. By the $\aleph_1$-completeness of $\mathbb{R}$, p is a condition in $\mathbb{R}$. Denote $\bigcup_{n\in\omega} \alpha_n$ by α. Since $r_{n+1} \leq p$ for every $n \in \omega$ we see that $p \Vdash \underset{\sim}{f} \upharpoonright \alpha = \bigcup_{n\in\omega} g_n$. Likewise, $p \Vdash \check{\alpha} \in \underset{\sim}{D}$ since $\underset{\sim}{D}$ is forced to be closed and $\alpha = \bigcup_{n\in\omega} \beta_n$.

Our goal is to show that the fixed g chosen above can serve as a weak diamond function from λ into 2. For this, we shall prove that the condition p can be extended to force $g(\alpha) = \underset{\sim}{c}(\underset{\sim}{f} \upharpoonright \check{\alpha})$. As $\underset{\sim}{c} \in V^{\mathbb{Q}}$, the value of $\underset{\sim}{c}(\underset{\sim}{f} \upharpoonright \check{\alpha})$ is determined by the condition p. However, functions from λ into 2 are not determined in a bounded stage. In particular, we can extend p to a condition q which forces $g(\alpha) = \underset{\sim}{c}(\underset{\sim}{f} \upharpoonright \check{\alpha})$. It follows that for every $\underset{\sim}{f} : \check{\lambda} \to 2$ and every club $\underset{\sim}{D}$ there exists an ordinal $\check{\alpha} \in \underset{\sim}{D}$ for which $\Vdash_{\mathbb{P}} g(\alpha) = c(f \upharpoonright \alpha)$, so we are done.

$\square_{1.7}$

The following diagram summarizes the relationship between the various prediction principles considered in this paper:



The downward positive implications $\Diamond_\lambda \Rightarrow \Phi_\lambda \Rightarrow \Psi_\lambda$ are trivial. The fact that Ψ_λ implies Galvin's property appears in [3], as well as the negative direction upwards (i.e. Galvin's property does not imply Ψ_λ). The consistency of Ψ_λ with $\neg\Phi_\lambda$ can be exemplified by a strongly inaccessible cardinal for which $\neg\Phi_\lambda$ is forced. The consistency of Φ_λ with $\neg\Diamond_\lambda$ can be forced by simple cardinal arithmetic considerations. Finally, it is shown in [1] that Galvin's property is not a theorem of ZFC, as its negation can be forced.

Finally, we mention the question of Shelah from [7], which seems to be the last open case:

Question 1.8. Assume λ is weakly inaccessible and $\langle 2^\theta : \theta < \lambda \rangle$ is not eventually constant. Is it consistent that $\neg\Phi_\lambda$ holds?

References

1. U. Abraham and S. Shelah, *On the intersection of closed unbounded sets*, J. Symbolic Logic **51** (1986), no. 1, 180–189. MR 830084 (87e:03117)
2. Keith J. Devlin and Saharon Shelah, *A weak version of $\diamondsuit$ which follows from $2^{\aleph_0} < 2^{\aleph_1}$*, Israel J. Math. **29** (1978), no. 2-3, 239–247. MR 0469756 (57 #9537)
3. Shimon Garti, *Weak diamond and Galvin's property*, forthcoming in Periodica Hungarica.
4. Moti Gitik, *Prikry-type forcings*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 1351–1447. MR 2768695
5. R. Björn Jensen, *The fine structure of the constructible hierarchy*, Ann. Math. Logic **4** (1972), 229–308; erratum, ibid. 4 (1972), 443, With a section by Jack Silver. MR 0309729 (46 #8834)
6. Lon Berk Radin, *Adding closed cofinal sequences to large cardinals*, Ann. Math. Logic **22** (1982), no. 3, 243–261. MR 670992
7. Saharon Shelah, *Middle diamond*, Arch. Math. Logic **44** (2005), no. 5, 527–560. MR 2210145 (2006k:03087)
8. ______, *Diamonds*, Proc. Amer. Math. Soc. **138** (2010), no. 6, 2151–2161. MR 2596054 (2011m:03084)

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