A model with a unique normal measure on κ and $2^\kappa = \kappa^{++}$ from optimal assumptions

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Abstract

Starting from a model with a measurable cardinal κ of Mitchell order $o(\kappa) = \kappa^{++}$, we construct a generic extension where $2^\kappa = \kappa^{++}$ and κ carries a unique normal measure. This answers a question of Friedman and Magidor from [5].

1 Introduction

In [5], Sy Friedman and Menachem Magidor introduced a forcing method for accurately extending the number of normal measures on a measurable cardinal κ. With their method, they were able to construct models of GCH together with any number of normal measures α , $1 \leq \alpha \leq \kappa^{++}$ from the minimal large cardinal assumption of a single measurable cardinal. Their result serves as the culmination point of a sequence results on the possible number of normal measures by Kunen [11], Kunen-Paris [12], Mitchell [14], Baldwin [2], Apter-Cummings-Hamkins [1], and Leaning [13] . In that paper, Friedman and Magidor further considered models where GCH fails and establish the consistency of a model with a measurable cardinal κ carrying a unique normal measure and $2^\kappa = \kappa^{++}$. The large cardinal assumption from their result is obtained is a $(\kappa + 2)-$strong cardinal[1], and the authors asked whether this large cardinal assumption can be lowered to $o(\kappa) = \kappa^{++}$, which is the lower bound by [9]. In [8], the second author of this paper constructed a model with a measurable cardinal κ and $2^\kappa = \kappa^{++}$ from this minimal assumption. This result is based on two forcing methods; an iteration adding a Cohen subset to regular cardinals $\alpha \leq \kappa$, and a second iteration of Prikry type forcings which adds a Prikry/Magidor/Radin generic sequence to each measurable cardinal below κ. Much like the Kunen-Paris model [12], these methods are known to produces what is consider a wild variety of normal measures on κ and cannot be directly used to answer the question in [5]. The purpose of this paper is to introduce an alternative to the construction of [8], allowing us to control the number of normal measures on κ and give an affirmative answer to the question.

Theorem 1. *The consitency strength of a measurable cardinal κ carrying a unique normal measure where $2^\kappa = \kappa^{++}$ is $o(\kappa) = \kappa^{++}$.*

Our forcing ,ethods $\mathbb{P}$ consists of three main parts, $\mathbb{P} = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$. Each $\mathbb{P}^i$, $i < 3$, is a nonstationary support iteration. $\mathbb{P}^0$ adds nonstationary Cohen functions $f_\alpha : \alpha \to \alpha$. $\mathbb{P}^1$ adds Prikry/Magidor/Radin generic sequences to measurable $\alpha < \kappa$. The iteration $\mathbb{P}^2$ combines generalized Sacks forcing, Collapse forcing, and a coding forcing. Our coding forcing combines the Friedman Magidor coding poset with a variant introduced by Cody and Eskew in [4].
A detailed description of the poset $\mathbb{P}$ and the proof of Theorem 1 is given in Section 2. The rest of this section is devoted to introduce the main forcing components and their basic properties. Our notations are standard and for the most part follow [8], [5], and [3]. We use the Jerusalem forcing convention by which a condition p extends a condition q (i.e., p is more informative than q) is denoted by $p \geq q$. We assume the reader is familiar with the basics of iteration of Prikry type forcings (i.e., see [8] and [10]) and with the basic theory of linear iteration (i.e., see, [15]).

[1]i.e., there exists an elementary embedding $j : V \to M$ with a critical point κ such that $V_{\kappa+2} \subset M$

1.1 A Nonstationary Support Iteration

Nonstationary support iteration were used in [5] and [3] to control the variety of normal measures appearing in generic extension. Let $\mathbb{P} = \langle \mathbb{P}_\alpha, \mathbb{Q}_\alpha \mid \alpha \leq \kappa \rangle$ be a nonstationary support iteration and suppose that for every $\alpha \leq \kappa$, $0_{\mathbb{P}_{\alpha+1}}$ forces $\mathbb{P}/G(\mathbb{P}_{\alpha+1})$ is a $(2^\alpha)^+$-closed forcing. A main feature of the nonstationary support of $\mathbb{P}$ is that if $j : V \to M$ is an ultrapower embedding by a normal measure on κ in V and $G \subset \mathbb{P}$ is generic, then the conditions in $j``G$ completely determine an $M[G]$-generic lter of $j(\mathbb{P}_\kappa)/G$. This is an immediate consequence of the following Lemma.

Lemma 2. *Suppose $d : \kappa \to V$ a function in V such that for every $\alpha < \kappa$, $d(\alpha)$ is a $\mathbb{P}_{\alpha+1}$ name for a dense open subset of $\mathbb{P}/G(\mathbb{P}_{\alpha+1})$. Then for every $p \in \mathbb{P}$ there are $q \geq p$ and a closed unbounded set $C \subset \kappa$ such that for every $\alpha \in C$, $q \restriction (\alpha+1) \Vdash q \setminus (\alpha+1) \in d(\alpha)$.*

For a proof, see [5] or [3].

1.2 A Nonstationary Variant of Cohen forcing

$\mathbb{P}^0 = \mathbb{P}^0_{\kappa+1} = \langle \mathbb{P}^0_\alpha, \mathbb{Q}^0_\alpha \mid \alpha \leq \kappa \rangle$ is a nonstatinoary support iteration of nonstationary Cohen posets. Let $\alpha \leq \kappa$ be an inaccessible cardinal in $V[G(\mathbb{P}^0_\alpha)]$. Conditions $p \in \mathbb{Q}^0_\alpha$ are partial functions $p : \alpha \to \alpha$ such that $\mathrm{dom}(p)$ is nonstationary in α, and $q \geq p$ if $p \subset q$. A generic filter of $G \subset \mathbb{Q}^0_\alpha$ is naturally identified with its induced generic function $f_\alpha = \bigcup_{p \in G} p$. We claim $\mathbb{Q}^0_\alpha$ preserves cardinals. It is easy to see $\mathbb{Q}^0_\alpha$ is $< \alpha$ closed and satisfies α^{++}.*c.c.* It remains to verify $\mathbb{Q}^0_\alpha$ preserves α^+.

Lemma 3. *$\mathbb{Q}^0_\alpha$ preserves α^+.*

Proof. Let $G^0_\alpha \subset \mathbb{P}^0_\alpha$ be generic over V, $p \in \mathbb{Q}^0_\alpha$, and $\dot\psi \in V[G^0_\alpha]$ be a $\mathbb{Q}^0_\alpha$ name for a function from α to α^+. Let us define four sequences.

1. An increasing sequence of conditions $\langle p_i \mid i < \alpha \rangle \subset \mathbb{Q}^0_\alpha$.
2. A $\subseteq$-decreasing sequence of closed unbounded subsets of α, $\langle C_i \mid i \leq \alpha \rangle$.
3. A continuous increasing sequence of ordinals $\langle \nu_i \mid i \leq \alpha \rangle \subset \alpha$.
4. A sequence of bounded sets $\langle x_i \mid i < \alpha \rangle \subset \mathcal{P}_\alpha(\alpha^+)$.

We maintain the following inductive assumptions.

1. $C_i \cap \mathrm{dom}(p_i) = \emptyset$ for all i.
2. $\nu_j \in C_i$ for all $i \leq j$.
3. $p_i \restriction \nu_i + 1 = p_j \restriction \nu_i + 1$ for all $i \leq j$.
4. for $j < \alpha$, $p_{j+1} \Vdash \Big(\dot f_\alpha[\nu_j + 1] \subset \nu_j + 1 \longrightarrow \dot\psi[\nu_j] \subset x_{j+1}\Big)$.

Set $p_0 = p$, $\nu_0 = 0$, and let $C_0 \subset \alpha$ be a closed unbounded set disjoint from $\mathrm{dom}(p_0)$. Suppose that $\langle p_i \mid i < \delta \rangle$, $\langle C_i \mid i < \delta \rangle$, and $\langle \nu_i \mid i < \delta \rangle$ have been defined. Let us define $p_\delta, C_\delta, \nu_\delta$, and x_δ.
Suppose $\delta = \gamma + 1$ is a successor ordinal. If $p_\gamma[\nu_\gamma + 1] \not\subset \nu_\gamma + 1$ then set $p_\delta = p_\gamma$, $x_\delta = x_\gamma$, $C_\delta = C_\gamma$, and $\nu_\delta = \min(C_\delta \setminus (\nu_\gamma + 1))$. Suppose that $p_\gamma[\nu_\gamma + 1] \subset \nu_\gamma + 1$. Since the partial function $p_\gamma \restriction \nu_\gamma + 1$ has only $\nu_\gamma^{\nu_\gamma} < \alpha$ many extensions $r : \nu_\gamma + 1 \to \nu_\gamma + 1$ and $\mathbb{Q}^0_\alpha$ is $< \alpha$ closed, we can find a partial function $t_\gamma : \alpha \setminus \nu_\gamma + 1 \to \alpha$ extending $p_\gamma \setminus \nu_\gamma + 1$ such that for every $r : \nu_\gamma + 1 \to \nu_\gamma + 1$, $r \cup t_\gamma \in \mathbb{Q}^0_\alpha$ decides the values $\dot\psi(\nu)$ for all $\nu < \nu_\gamma$. Namely, there are ordinals $\eta(r, \nu) < \alpha$ so that $r \cup t_\gamma \Vdash \dot\psi(\check\nu) = \eta(\check r, \nu)$. Define $x_\delta = \{\eta(r, \nu) \mid \nu < \nu_\gamma, r \in (\nu_\gamma + 1)^{\nu_\gamma + 1}\}$, $p_\delta = p_\gamma \restriction (\nu_\gamma + 1) \cup t_\gamma$ and take $C_\delta \subset C_\gamma$ be a closed unbounded subset of α disjoint from $\mathrm{dom}(p_\delta)$. Suppose now that $\delta \leq \alpha$ is a limit ordinal and let $\nu_\delta = \sup_{i<\delta} \nu_i$. We then set $x_\delta = \emptyset$, $C_\delta = \bigcap_{i<\delta} C_i$ if $\delta < \alpha$ and $C_\delta = \Delta_{i<\alpha} C_i$ if $\delta = \alpha$, and $p_\delta = \bigcup_{i<\delta} p_i$. Finally, let $p^* = p_\alpha$ and $X = \bigcup_{i<\alpha} x_i$. Then $p^* \geq p$ and $X \in [\alpha^+]^\alpha$. We claim that $p^* \Vdash \mathrm{rng}(f) \subset X$. It is sufficient to verify that for every $\nu < \alpha$, the set $\{q' \geq p^* \mid q' \Vdash \dot\psi[\nu] \subset X\}$ is a dense subset of $\mathbb{Q}^0_\alpha$. Fix $\nu < \alpha$ and $q \geq p^*$. There exists $\gamma < \alpha$ for which $\nu_\gamma \geq \nu$, $\nu_\gamma \notin \mathrm{dom}(q)$, and $\mathrm{rng}(q \restriction \nu_\gamma) \subset \nu_\gamma$. Let $r : \nu_\gamma + 1 \to \nu_\gamma + 1$ be any completion of $q \restriction \nu_\gamma + 1$. Then $q' = r \cup q$ is an extension of $p_{\gamma+1}$ which forces $\dot f_\alpha[\nu_\gamma + 1] \subset \nu_\gamma + 1$. Hence $q' \Vdash \dot\psi[\nu_\gamma] \subset \check X$. □

The arguments of [5] assert the nonstationay support iteration $\mathbb{P}^0$ preserves all cardinals and stationary sets assuming each of the posets $\mathbb{Q}^0_\alpha$ do.

Lemma 4.

1. *Suppose α is a strongly inaccessible cardinal. Let $\dot\phi$ be a $\mathbb{Q}^0_\alpha$-name of a function from $\alpha^n\alpha$ for some $n<\omega$, and $h:\alpha\to\alpha$. Then for every $p\in\mathbb{Q}^0_\alpha$ there are $q\geq p$, $C\subset\alpha$ closed unbounded, and $F:C\to[\alpha]^{<\alpha}$ with $|F(\nu)|\leq\max(2^\nu,|h(\nu)|)$ for all $\nu\in C$, such that*
$$q\Vdash\forall\nu\in\check{C}.\left(\dot{f}_\alpha[\nu+1]\subset\nu+1\right)\longrightarrow\left(\dot\phi[\check{h}(\nu)^n]\subset\check{F}(\nu)\right).$$
2. *Suppose $d:\alpha\to V$ is a function satisfying $d(\nu)\subset\mathbb{Q}^0_\alpha$ is dense open for every $\nu<\alpha$. Then for every function $b:\alpha\to\alpha$ and $p\in\mathbb{Q}^0_\alpha$ there are $q\geq p$ and a closed unbounded set $C\subset\kappa$ so that q forces that for all $\nu\in C$, if $\dot{f}_\alpha[\nu+1]\subset b(\nu)+1$ then $(f_\alpha\restriction\nu+1)\cup q\in d(\nu)$.*
3. *$\mathbb{Q}^0_\alpha$ preserves all stationary subsets of α^+ in $V^{\mathbb{P}^0_\alpha}$.*

Proof. The proof of the first two parts is similar to the proof of the preceding Lemma 3. Let us prove the third assertion. Suppose that $G^0_\alpha\subset\mathbb{P}^0_\alpha$ is a V-generic filter and $S\subset\alpha^+\cap\operatorname{cf}(\alpha)$ is a stationary subsets of α^+. Let $\dot{C}$ be a $\mathbb{Q}^0_\alpha$ name of a closed unbounded subset of α^+ and $p\in\mathbb{Q}^0_\alpha$. Fix a sufficiently large regular $\theta>2^{\alpha^+}$ and let $N\prec H_\theta[G^0_\alpha]$ be an elementary substructure satisfying, $|N|=\alpha$, $N^{<\alpha}\subset N$, $N\cap\alpha^+=\tau\in S$, and $p,\dot{C}\in N$. Let $\vec\tau=\langle\tau_i\mid i<\alpha\rangle$ be a cofinal increasing sequence in τ. Since N contains all initial segments of $\vec\tau$, it is possible to construct sequences $\langle p_i\mid i<\alpha\rangle$, $\langle\nu_i\mid i<\alpha\rangle$, $\langle C_i\mid i<\alpha\rangle$, $\langle x_i\mid i<\alpha\rangle$ whose initial segments belong to N and $x_i\in\mathcal{P}_\alpha(\tau)$ for all $i<\alpha$, such that for every $j<\alpha$,
$$p_{j+1}\Vdash(\dot{f}_\alpha[\nu_j+1]\subset\nu_j+1\implies\forall i<j.\min(C\setminus\check\tau_i+1)\in x_j).$$
Let $p^*=p_\alpha$ be an upper limit of all p_i, $i<\alpha$. Then p^* forces τ is a limit point of $\dot{C}$, thus $p^*\Vdash\check{S}\cap\dot{C}\neq\emptyset$. □

Let $G^0\subset\mathbb{P}^0$ be a V-generic filter. The proof of the main Theorem 1 relies on an analysis of extensions of ground model embeddings $i:V\to M$ with $\operatorname{cp}(i)=\kappa$, in $V[G^0]$. The following Lemma describes the possible extension when i,M correspond to a V-ultrapower by a normal measure on κ.

Lemma 5. *Suppose that $i:V\to M\cong\operatorname{Ult}(V,U)$ is an ultrapower embedding of V by a normal measure U on κ. For every $\beta<i(\kappa)$ there exists a unique M-generic filter $H^{0,\beta}\subset i(\mathbb{P}^0)$ satisfying the following properties.*

- *$H^{0,\beta}\cap(\mathbb{P}^0_{\kappa+1})^M=G^0$.*
- *$i``G^0\subset H^{0,\beta}$.*
- *The $\mathbb{Q}^0_{i(\kappa)}$ generically induced function $f^{H^{0,\beta}}_{i(\kappa)}=\bigcup\{p(i(\kappa))\mid p\in H^{0,\beta}\}:i(\kappa)\to i(\kappa)$ satisfies $f^{H^{0,\beta}}_{i(\kappa)}(\kappa)=\beta$.*

Proof. $G^0=G^0_{\kappa+1}\subset\mathbb{P}^0_{\kappa+1}$ is a $i(\mathbb{P}^0)\restriction\kappa+1$ generic filter over M. Denote $G^0\cap\mathbb{P}^0_\kappa$ by G^0_κ. By Lemma 2, $i``G^0_\kappa$ generates a generic filter for $i(\mathbb{P}^0_\kappa)/G^0$ over $M[G^0]$. Denote the resulting generic by $H^0_{i(\kappa)}$. It follows that $i:V\to M$ extends to $i^*:V[G^0_\kappa]\to M[H^0_{i(\kappa)}]$. Denote $\{q(\kappa)\in\mathbb{Q}^0_\kappa\mid q\in G^0\}$ by $G^0(\kappa)$ and let $h^*=\bigcup i^*``G^0(\kappa)$. We claim $h^*:i(\kappa)\to i(\kappa)$ is a partial function with $\operatorname{dom}(h^*)=i(\kappa)\setminus\{\kappa\}$. If is easy to see that $h^*\restriction\kappa=f^{G^0}_\kappa:\kappa\to\kappa$ and that $\kappa\notin\operatorname{dom}i^*(p)$ for every $p\in G^0(\kappa)$. Let $\gamma\in(\kappa,i(\kappa))$ then $\gamma=i(h)(\kappa)$ for some $h:\kappa\to\kappa$ in V. Let $C_h=\{\nu<\kappa\mid h[\nu]\subset\nu\}$. By a standard density argument, there is $p\in G^0(\kappa)$ such that $\kappa\setminus C_h\subset\operatorname{dom}(p)$. Thus $\gamma\in\operatorname{dom}(i^*(p))\subset\operatorname{dom}(h^*)$. For every $\beta<i(\kappa)$ let $h^\beta=h^*\cup\{\langle\kappa,\beta\rangle\}$. It remains to verify h^β is generic. Let $D\subset i^*(\mathbb{Q}^0_\kappa)$ be a dense open set. There are functions b,d with domain κ such that $\beta=i(b)(\kappa)$ and $D=i^*(d)(\kappa)$. By the second part of Lemma 4, there is $q\in G^0(\kappa)$ such that
$$i^*(q)\Vdash\left(\dot{f}_{i(\kappa)}[\kappa+1]\subset i^*(b)(\kappa)+1\longrightarrow(\dot{f}_{i(\kappa)}\restriction\kappa+1)\cup i^*(q)\in i^*(d)(\kappa)\right).$$
It follows that $h^\beta\restriction(\kappa+1)\cup i^*(q)\in D$. □

1.3 A Coding Poset

One of the key ingredients in the forcing construction of [5] is the coding posets, Code_α, $\alpha \leq \kappa$, which is used for coding information guaranteeing the uniqueness of a certain generic object. For each nontrivial forcing stage $\alpha \leq \kappa$, the coding poset Code_α of [5], is designed to code subsets of α^+ or α^{++}. Our intention is to code larger subsets, of size $f(\alpha)$, where $f = f_\kappa : \kappa \to \kappa$ is a nonstationary Cohen generic function. For this we appeal to the coding poset of [4], which is more flexibility in terms of the size of information. We will define this coding poset, for coding a set X using a sequence of almost disjoint stationary sets $\vec{T}$, which will be denote by $\mathrm{Code}_{\vec{T}}(X)$. All assertions stated in this Section which concern our coding posets are an immediate consequence of the arguments of [5] and [4].

Definition 6 (A $\blacklozenge_{\alpha^+,\delta}$ sequence)**.** Let α be a cardinal and $\delta > \alpha^+$ an ordinal. A sequence $\vec{d}_{\alpha^+,\delta} = \{d_z \mid z \in [\delta]^{<\alpha^+}\}$ is a $\blacklozenge_{\alpha^+,\delta}$ sequence if for every subset $A \subset \delta$ and every closed unbounded set $C \subset [\delta]^{<\alpha^+}$, such that the set $\{z \in [\delta]^{<\alpha^+} \mid d_z = A \cap z\}$ contains $\subset$-increasing and closed sequences of any length $\beta < \alpha^+$.

Let $g : \alpha^+ \to 2$ be an $\mathrm{Add}(\alpha^+)$ generic function. For every $\delta > \alpha^+$ define a sequence $\vec{d}_{\alpha^+,\delta} = \langle d_z \mid z \in [\delta]^{<\alpha^+}\rangle$ as follows. For every $z \in [\delta]^{<\alpha^+}$ let $o_z : \mathrm{otp}(z) \to z$ be an order preserving enumeration of z. Let $d_z = \{o_z(\beta) \mid g(\sup(z \cap \alpha^+) + \beta) = 1\}$ [2]. The sequence $\vec{d}_{\alpha^+,\delta} = \{d_z \mid z \in [\delta]^{<\alpha^+}\}$ is a $\blacklozenge_{\alpha^+,\delta}$ sequence. Furthermore, the fact $\vec{d}_{\alpha^+,\delta}$ is $\blacklozenge_{\alpha^+,\delta}$ is preserved in all $< \alpha^+$-closed forcing extensions.

Notation 7. *Fix a stationary set $T \subset \alpha^+ \cap \mathrm{cf}(\alpha)$. For every $\nu < \delta$, set $T^\delta_\nu = \{z \in [\delta]^{<\alpha^+} \mid z \cap \alpha^+ \in T, d_z = \{\nu\}\}$. For every $\eta > \alpha^+$, let $\vec{T}^\eta = \{T^\delta_\nu \mid \nu < \delta < \eta\}$.*

Note that for every $\delta' < \delta$ and $\nu \neq \nu'$, then the sets $\pi_{\delta,\delta'}(T^\delta_\nu) = \{z \cap \delta' \mid z \in T^\delta_\nu\}$, and $T^{\delta'}_{\nu'}$ are almost disjoint stationary subsets of $[\delta']^{<\alpha^+}$.

We turn to define the coding poset $\mathrm{Code}_{\vec{T}}(X)$ for coding a set of ordinals $X \subset \eta$ for some regular cardinal η, using a sequence of almost disjoint stationary sets $\vec{T}$. The coding poset is defined in a generic extension of the universe in which η is collapsed to become α^+. We first set-up the ground model assumption. Suppose that g is an $\mathrm{Add}(\alpha^+)$ generic and that $\mathbb{Q} \in V[g]$ is a α^+-closed forcing collapsing all cardinals $< \eta$ to α. Let $H \subset \mathbb{Q}$ be a generic filter over $V[g]$. Working in $V[g * H]$, we fix for each $\delta \in [\alpha^+, \eta)$ an increasing, cofinal, and continuous sequence $\vec{z}(\delta) = \langle z^\delta_i \mid i < \alpha^+\rangle$ in $[\delta]^{<\alpha^+}$.

Notation 8. *For a subset $c \subset \alpha^+$, denote $\{z^\delta_i \mid i \in c\}$ by $\vec{z}(\delta) \restriction c$.*

Definition 9 ($\mathrm{Code}_{\vec{T}}(X)$)**.** Working in the extension $V[g * H]$ described above, let $\mathrm{Code}_{\vec{T}}(X)$ be a $\leq \alpha$ support product of forcings which add closed unbounded subsets to $\vec{z}(\delta)$, for all $\delta \in [\alpha^+, \eta)$. Conditions $q \in \mathrm{Code}_{\vec{T}}(X)$ are partial functions $q : \alpha^+ \to [\alpha]^{<\alpha}$ of size $|q| \leq \alpha$ so that for each $\nu \in \mathrm{dom}(q)$, $q(\nu)$ is a closed bounded subset of α^+ satisfying that $\vec{z}(\alpha^+ \cdot \nu + \alpha^+) \restriction q(\nu) \cap T^{\alpha^+\cdot\nu+\alpha^+}_{\alpha^+\cdot\nu+0} = \emptyset$ if $\nu \in X$, and $\vec{z}(\alpha^+ \cdot \nu + \alpha^+) \restriction q(\nu) \cap T^{\alpha^+\cdot\nu+\alpha^+}_{\alpha^+\cdot\nu+1} = \emptyset$ if $\nu \notin X$.

Let $q, p \in \mathrm{Code}_{\vec{T}}(X)$, then q extends p (denoted $q \geq p$) if the following hold.

- $\mathrm{dom}(p) \subset \mathrm{dom}(q)$,
- for every $\nu \in \mathrm{dom}(p)$, $q(\nu)$ is an end extension of $p(\nu)$,
- $$\vec{z}(\alpha^+ \cdot \nu + \alpha^+) \restriction (q(\nu) \setminus p(\nu)) \cap T^{\alpha^+\cdot\nu+\alpha^+}_{\alpha^+\cdot\nu+\alpha^++2\cdot(1+\tau)+0} = \emptyset \text{ if } \tau \in p(\nu)$$

  and
  $$\vec{z}(\alpha^+ \cdot \nu + \alpha^+) \restriction (q(\nu) \setminus p(\nu)) \cap T^{\alpha^+\cdot\nu+\alpha^+}_{\alpha^+\cdot\nu+\alpha^++2\cdot(1+\tau)+1} = \emptyset \text{ if } \tau \notin p(\nu).$$

Lemma 10. $\mathrm{Code}_{\vec{T}}(X)$ *satisfies the following properties.*

1. $\mathrm{Code}_{\vec{T}}(X)$ *is η.c.c, $< \alpha$-closed, and $< \alpha^+$-distributive.*
2. *If $W \subset [\delta]^{<\alpha^+}$ is stationary and $\pi_{\delta,\alpha^+}(W)$ is almost disjoint from T, then W remains stationary in any* $\mathrm{Code}_{\vec{T}}(X)$ *generic extension.*
3. *Let $G \subset$* $\mathrm{Code}_{\vec{T}}(X)$ *be a generic filter over $V^{\mathrm{Add}(\alpha^+)*\mathbb{Q}}$. The following hold in $V[g * H * G]$.*

[2]Note that for closed unbounded many $z \in [\delta]^{<\alpha^+}$, $z \cap \alpha^+ \in \alpha^+$, thus $d_z = \{o_z(\beta) \mid g((z \cap \alpha^+) + \beta) = 1\}$.

- *For every $\nu < \eta$,*

$$\nu \in X \iff T^{\alpha^+\cdot\nu+\alpha^+}_{\alpha^+\cdot\delta+2+\beta+0} \text{ is nonstationary } \iff T^{\alpha^+\cdot\delta+\alpha^+}_{\alpha^+\cdot\delta+2+\beta+1} \text{ is stationary .}$$

- *For each $\nu < \eta$, let $C(\nu) = \bigcup\{p(\nu) \mid p \in G\} \subset \alpha^+$. $C(\nu)$ is closed unbounded and satisfies that for every $\beta < \alpha^+$,*

$$\beta \in C(\nu) \iff T^{\alpha^+\cdot\nu+\alpha^+}_{\alpha^+\cdot\nu+2+\beta+0} \text{ is nonstationary } \iff T^{\alpha^+\cdot\nu+\alpha^+}_{\alpha^+\cdot\nu+2+\beta+1} \text{ is stationary .}$$

Corollary 11. *Suppose that $X = X(H) \in V[g * H]$ is a subset of η encoding the generic H via some fixed simple encoding recipe. Suppose that $G \subset \mathrm{Code}_{\vec{T}}(X)$ be generic over $V[g * H]$. then (H, G) is the unique pair (H', G') so that $H' \subset \mathbb{Q}$ is a $V[g]$ generic, and $G' \subset \mathrm{Code}_{\vec{T}}(X')$ is $V[g * H']$ generic coding $X' = X(H') \subset \eta$.*

Note that the last Corollary does not completely determine the generic filters added to V, as it does not determine the generic Cohen function $g \in {}^{\alpha^+}2$. The difficulty in extending the coding to include g is that g determines the stationary sets of $\vec{T}$ which serve as the infrastructure of the coding poset. To overcome this difficulty, let us incorporate an additional independent coding poset for coding $g : \alpha^+ \to 2$ using a canonically defined sequence of pairwise almost disjoint subsets of $\alpha^+ \cap \mathrm{cf}(\alpha)$. Suppose that $V = L[\vec{U}]$ is a Mitchell model. Let $\vec{S} = \langle S_i \mid i < \alpha^+\rangle$ be a sequence of almost disjoint stationary subsets of $\alpha^+ \cap \mathrm{cf}(\alpha)$, derived from a minimally definable diamond sequence over $H^V_{\alpha^+}$. Suppose that $\mathbb{P}$ is a forcing which preserves all cardinals $\geq \alpha$ and stationary subsets of α^+. Let $G(\mathbb{P}) \subset \mathbb{P}$ be a generic filter over V, g, an $\mathrm{Add}(\alpha^+)$ generic over $V[G(\mathbb{P})]$, and $G(\mathrm{Col}) \subset \mathbb{Q} = \mathrm{Col}(\alpha^+, <\eta)$ be a Levy generic filter over $V[G(\mathbb{P}) * g]$, where $\eta \geq \alpha^+$ is regular [3]. Since $\mathbb{P}$ preserves all stationary subsets of α^+, the sets in $\vec{S}$ are stationary in $V[G(\mathbb{Q}) * G(\mathrm{Col})]$. Working in $V[G(\mathbb{P}) * g * G(\mathrm{Col})]$, we define the two steps coding poset $\mathrm{Code}(\alpha, X)$ for coding a subset $X \subset \eta$.

Definition 12 (A two steps coding poset)**.** $\mathrm{Code}(\alpha, X) = \mathrm{Code}_0(\alpha) * \mathrm{Code}_1(\alpha, X)$ is a two-steps iteration. Let $\mathrm{Code}_0(\alpha) = \mathrm{Code}_{\vec{S}\setminus 1}(X_0)$, where $\vec{S} \setminus 1 = \langle S_i \mid 1 \leq i < \alpha^+\rangle$ and $X_0 \subset \alpha^+$ is a simple encoding of g. Let $G_0 \subset \mathrm{Code}_0(\alpha)$ be $V[G(\mathbb{Q}) * g * G(\mathrm{Col})]$ and $T = S_0$. By Lemma 10, all stationary sets in $\vec{T}$ remain stationary in $V[G(\mathbb{Q}) * g * G(\mathrm{Col}) * G_0]$. Define $\mathrm{Code}_1(\alpha, X) = \mathrm{Code}_{\vec{T}}(X)$.

The subset $X \subset \eta$ can represent (via some simple encoding recipe) any information of size η. Assuming X includes an encoding of $G(\mathbb{P}) * G(\mathrm{Col})$, the structure of $\mathrm{Code}(\alpha, X)$ guarantees the generic uniqueness of $G(\mathbb{P}) * g * G(\mathrm{Col}) * \mathrm{Code}(\alpha, X)$.

Corollary 13. *Let $X = X\left(G(\mathbb{P}) * g * G(\mathrm{Col})\right) \subset \eta$ be a fixed encoding of the generic filters $G(\mathbb{P}) * g * G(\mathrm{Col}) \subset \mathbb{P} * \mathrm{Add}(\alpha^+) * \mathrm{Col}(\alpha^+, <\eta)$. Suppose that $G(\mathrm{Code}) \subset \mathrm{Code}(\alpha, X)$ is generic over $V[G(\mathbb{P}) * g * G(\mathrm{Col})]$, then $G(\mathbb{P}) * g * G(\mathrm{Col}) * G(\mathrm{Code})$ is the only V generic filter in $V[G(\mathbb{Q}) * g * G(\mathrm{Col}) * G(\mathrm{Code})]$.*

1.4 Generalized Sacks forcing its iterations

The effect of a generalized Sacks forcing on possible generic extensions of ground model ultrapower embeddings was initially studied in [6]. In [5], the authors use the following version to blow up the power set of κ while obtaining a unique normal measure on κ.

Definition 14. Let $\mathrm{Sacks}^*(\alpha)$ denote the poset which consists of all subtrees $T \subset 2^{<\alpha}$ which are closed under increasing sequences of length less than α, and for which the set Split_T of all ordinals $\beta < \alpha$ satisfying that every node $t \in T$ of height β splits to $t^\frown\langle 0\rangle, t^\frown\langle 1\rangle \in T$, is of the form $\mathrm{Split}_T = C \cap \mathrm{Sing}$ where $C \subset \alpha$ is a closed unbounded set, and Sing is the set of all singular ordinals. For $\eta \geq \alpha$, $\mathrm{Sacks}^*(\alpha, \eta)$ denotes the $\leq \alpha$ support product of η-many copies of $\mathrm{Sacks}^*(\alpha)$.

It is shown in [6] and [5] that $\mathrm{Sacks}^*(\alpha, \eta)$ satisfies $\alpha^{++}.c.c$, it is $< \alpha$-closed and admits a closure property for suitable fusion sequences of trees of length α. The last implies α^+ is preserved, as well as all stationary subsets of α^+. It is shown in [5] that if $j : V \to M$ is an extender embedding which (with a suitable preceding iteration) allows j to extend in a generic forcing extension including $\mathrm{Sacks}^*(\alpha, \eta)$, then the j pointwise image of the Sacks^* generic, completely determines a generic filter for $j(\mathrm{Sacks}^*(\alpha, \eta))$ on the M side.

[3] We can also think of g as extracted from $G(Col)$ which subsumes $\mathrm{Add}(\alpha^+)$.

Let $\mathbb{P}^2 = \langle \mathbb{P}^2_\alpha, \mathbb{Q}^2_\alpha \mid \alpha \leq \kappa \rangle$ a nonstationary support iteration such that $\alpha \leq \kappa$ is a nontrivial iteration stage if and only if it is an inaccessible cardinal, and then $\mathbb{Q}^2_\alpha = \text{Sacks}^*(\alpha, f(\alpha)) * \text{Add}(\alpha^+) * \text{Col}(\alpha^+ < f(\alpha)) * \text{Code}(\alpha, X_\alpha)$, Where $f : \kappa + 1 \to \kappa + 1$ maps all ordinals to regular cardinals, $\text{Col}(\alpha^+, < f(\alpha))$ is the Levy collapse, and $\text{Code}(\alpha, X_\alpha)$ codes $G^2 \restriction \mathbb{P}^2_\alpha * \text{Sacks}^*(\alpha, f(\alpha)) * \text{Add}(\alpha^+) * \text{Col}(\alpha^+ < f(\alpha))$. The arguments of [5] and Corollary 13 above imply $\mathbb{P}^2$ satisfies the following properties.

Lemma 15. *Let $G^2 \subset \mathbb{P}^2$ be a generic filter. Then the following holds in $V[G^2]$.*

1. *$\mathbb{P}^2$ preserves all cardinal $\beta \leq \kappa$ which do not belong to an interval $(\alpha^+, f(\alpha))$ for some nontrivial iteration stage α of $\mathbb{P}^2$.*
2. *For each inaccessible $\alpha \leq \kappa$, $\mathbb{P}^2_\alpha$ preserves all stationary subsets of α^+.*
3. *G^2 is the unique V-generic filter for $\mathbb{P}^2$ in $V[G^2]$.*
4. *For every function $g : \kappa \to \kappa$ in $V[G^2]$ there is a function $g_0 : \kappa \to [\kappa]^{<\kappa}$ in V satisfying $|g_0(\alpha)| \leq f(\alpha)$ such that $g(\alpha) \in g_0(\alpha)$ for all $\alpha < \kappa$.*

Lemma 16. *Suppose that $G^2 \subset \mathbb{P}^2$ is generic over the ground model $V = L[E]$ and $j : V \to M \cong \text{Ult}(V, U)$ is an ultrapower embedding by a $(\kappa, f(\kappa))$-extender on κ in V, then j has a unique extension $j^* : V[G^2] \to M[H^2]$ in $V[G^2]$.*

2 The Model

This section is devoted to defining the main forcing notion $\mathbb{P} = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ and proving Theorem 1. We force over a Mitchell model $V = L[\vec{U}]$, where $\vec{U} = \langle U_{\alpha,\tau} \mid \alpha \leq \kappa, \tau < o(\alpha) \rangle$ is a coherent sequence of normal measures with $o(\kappa) = \kappa^{++}$. For each $\alpha \leq \kappa$ and $\tau < o(\alpha)$, let $i_{\alpha,\tau} : V \to M_{\alpha,\tau}$ denote the ultrapower embedding by $U_{\alpha,\tau}$.
Our goal is to construct a generic extension in which $2^\kappa = \kappa^{++}$ and κ carries a unique normal measure. The V elementary embedding which will correspond to the measure ultrapower of the generic extension is denoted by j. We describe j by factoring it into three parts $j = i' \circ k \circ i$. First, $i : V \to N$ is a direct limit embedding of an iterated ultrapower of length κ_V^{++} by the measures in $\vec{U}$ and its images. $\text{cp}(i) = \kappa$, and the choice of critical points and measures constructing i, is desgined to guarantee that the critical points of this iteration generate generic Prikry/Magidor/Radin to all measurable cardinals in M, between κ and $i(\kappa)$. We have $i(\kappa) = \kappa^{++}$. Second, $k : N \to N' \cong \text{Ult}(N, i(U_{\kappa,0}))$ is the ultrapower embedding by the measure $i(U_{\kappa,0})$ on $i(\kappa)$. Finally, $i' : N' \to M$ results from an iteration similar to i, using measures on ordinals strictly between $i(\kappa)$ and $k(i(\kappa))$. It follows that $j(\kappa) = k(i(\kappa)) > i(\kappa) = \kappa^{++}$.
Now, the goal of forcing with $\mathbb{P} = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ is to obtain a generic extension in which $j : V \to M$ is the unique ground model embedding which extends to a normal measure ultrapower embedding. This extension will be denoted by j^2. Achieving such an extension of j requires the poset $\mathbb{P} = \mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2$ to add the following sets and functions.

- $\mathbb{P}$ needs to add many sequences of ordinals, so that $j(\mathbb{P})$ generates an extension of M which is closed under κ sequences[4]. The component of $\mathbb{P}$ responsible for this is $\mathbb{P}^1$. It is an iteration of Prikry/Magidor/Radin forcings, adding cofinal sequences to all measurable cardinals $\alpha < \kappa$, according to their Mitchell order.
- For every generator (critical point) γ, of the iteration leading to j, $\mathbb{P}$ needs to add a function $h : \kappa \to \kappa$ such that $j^2(h)(\kappa) = \gamma$. All three components $\mathbb{P}^0, \mathbb{P}^1, \mathbb{P}^2$ come into play here. The poset $\mathbb{P}^0$ adds a function which captures the generator $\kappa_V^{++} = \text{cp}(k)$. $\mathbb{P}^0$ is an iteration of nonstationary Cohen posets. The function $f = f_\kappa : \kappa \to \kappa$ added at the last stage of the iteration, will satisfy $j^2(f)(\kappa) = \kappa_V^{++}$ [5]. The poset $\mathbb{P}^2$ adds κ^{++} many generalized Sacks function, capturing all generators $\gamma < \kappa_V^{++}$. To capture the generators $\gamma \in (\kappa_V^{++}, j(\kappa))$, the poset $\mathbb{P}^1$ is also required. The key is that every generator in this interval must appear as Prikry/Magidor/Radin point on a generic sequence for some ordinal $\delta < j(\kappa)$ which can be represented as $\delta = j(h)(\vec{\tau})$, where $h \in V$ and $\vec{\tau}$ is a finite sequence of generators on ordinals $\leq \kappa_V^{++}$.

Finally, we want all the generic information mentioned above to be added in a "tame" manner, allowing only a single generic extension of $j(\mathbb{P})$ to be compatible with the pointwise image $j``G(\mathbb{P})$ of a V generic

[4]Note that M is not even closed under ω sequences of ordinals.
[5]Note that κ_V^{++} is an inaccessible cardinal in M.

$G(\mathbb{P}) \subset \mathbb{P}$. All parts $\mathbb{P}^0, \mathbb{P}^1\mathbb{P}^2$ are designed to guarantee this. Most importantly, we use the nonstationary supports of all parts, and the features of the nonstationary Cohen, Sacks, Collapse, and Prikry type posets, to argue that the $j``G(\mathbb{P})$ completely determine substantial parts of a $j(\mathbb{P})$ generic filter. Then, to fix the parts of $j(\mathbb{P})$ which are not covered by the j pointwise image of $G(\mathbb{P})$, we will use the coding poset described in the previous Section.

We proceed to define the posets $\mathbb{P}^0$,$\mathbb{P}^1$,$\mathbb{P}^2$. $\mathbb{P}^0 = \mathbb{P}^0_{\kappa+1} = \langle \mathbb{P}^0_\alpha, \mathbb{Q}^0_\alpha \mid \alpha \leq \kappa\rangle$ is the nonstationary support iteration of nonstationary Cohen posets, which was introduced in Section 1.2. Let $G^0 \subset \mathbb{P}^0$ be a generic filter, and for each inaccessible $\alpha \leq \kappa$, let $f^{G^0}_\alpha = \bigcup\{p(\alpha) \mid p \in G^0, \alpha \in \mathrm{supp}(p)\}$. Fix $\alpha \leq \kappa$ and $\tau < o(\alpha)$. Lemma 5 asserts that for each $\beta < i_{\alpha,\tau}(\alpha)$, the embedding $i_{\alpha,\tau} : V \to M_{\alpha,\tau}$ has a unique extension $i^\beta_{\alpha,\tau} : V[G^0] \to M_{\alpha,\tau}[H^{0,\beta}_{\alpha,\tau}]$ in $V[G^0]$ satisfying the following properties.

1. $i_{\alpha,\tau}``G^0_{\alpha+1} \subset H^{0,\beta}_{\alpha,\tau}$.
2. $f^{H^{\alpha,\beta}_{\alpha,\tau}}_{i_{\alpha,\tau}(\alpha)}(\alpha) = \beta$.

We define $U^\beta_{\alpha,\tau}$ to be the normal measure on α derived from $i^\beta_{\alpha,\tau}$.
The next stage of our construction is based on the coherent sequence $\vec{U}^0 \restriction \kappa = \langle U^0_{\alpha,\tau} \mid \alpha < \kappa, \tau < o(\alpha)\rangle$ in $V[G^0]$. Note that for every measurable $\alpha < \kappa$, $U^0_{\alpha,\tau}$ contains the set $\{\delta < \alpha \mid f_\alpha(\delta) = 0\}$.
$\mathbb{P}^1 = \mathbb{P}^1_\kappa = \langle \mathbb{P}^1_\alpha, \mathbb{Q}^1_\alpha \mid \alpha < \kappa\rangle$ is a nonstationary support iteration of Prikry type forcings, adding Prikry/Magidor/Radin sequences to all measurable cardinals $\alpha < \kappa$, using the measures in the coherent sequence $\vec{U}^0 \restriction \kappa = \{U^0_{\alpha,\tau} \mid \alpha < \kappa, \tau < o(\alpha)\}$. An iteration of Prikry type forcings for changing cofinalities was introduced in [7] and further studied in [8]. A nonstationary support variant of the iteration was studied in [3]. Conditions $p \in \mathbb{P}^1$ are partial functions with $\mathrm{dom}(p) \subset \kappa$ is nonstationary below each Mahlo cardinal $\alpha \leq \kappa$. For each $\alpha \in \mathrm{dom}(p)$, $p(\alpha)$ is a $\mathbb{P}_\alpha$ name of a condition in a Prikry type forcing $\mathbb{Q}_\alpha$ which adds a Prikry/Magidor/Radin cofinal sequence to α according to $o(\alpha)$. Given $p, q \in \mathbb{P}$, p extends q (denoted $p \geq q$) if $\mathrm{dom}(q) \subset \mathrm{dom}(p)$, $p \restriction \alpha \Vdash p(\alpha) \geq_{\mathbb{Q}_\alpha} q(\alpha)$ for all $\alpha \in \mathrm{dom}(q)$, and $p \restriction \alpha \Vdash p(\alpha) \geq^*_{\mathbb{Q}_\alpha} q(\alpha)$ (i.e., $p(\alpha)$ is a direct extension of $q(\alpha)$ for all but finitely many $\alpha \in \mathrm{dom}(q)$. Lemma 2 guarantees that for every $p \in \mathbb{P}^1$ and $d : \kappa \to V[G^0]$ such that each $d(\alpha)$ is a name for a $\leq^*$ dense open subset of $\mathbb{P}^1/G(\mathbb{P}^1_{\alpha+1})$, there are $p^* \geq^* p$ and a closed unbounded set $C \subset \kappa$ such that for every $\alpha \in C$, $q \restriction (\alpha + 1) \Vdash q \setminus (\alpha + 1) \in d(\alpha)$. Let $G^1 \subset \mathbb{P}^1$ be a generic filter over $V[G^0]$. We now follow the construction of [8] to define an elementary embedding j^1 of $V[G^0 * G^1]$. For each $\tau < \kappa^{++}$, the measure $U^0_{\kappa,\tau} \in V[G^0]$ extends in $V[G^0 * G^1]$ to a κ complete ultrafilter, denoted in [8] by $U^*_{\kappa,\tau}(\emptyset)$, and the sequence $\langle U^*_{\kappa,\tau}(\emptyset) \mid \tau < \kappa^{++}\rangle$ is a Rudin-Keisler (RK) increasing sequence.
Let $i^1 : V[G^0 * G^1] \to \bar{M}^1$ denote the direct limit embedding of the RK-direct system. Then $i^1(\kappa) = \kappa^{++}$. To complete the construction, an additional ultrapower is required. Let $k^1 : \bar{M}^1 \to \tilde{M}^1 \cong \mathrm{Ult}(\bar{M}^1, i^1(U^1_0))$ be the ultrapower embedding of $\bar{M}^1$ by $i^1(U^1_0)$ and $\tilde{j}^1 = k^1 \circ i^1 : V[G^0 * G^1] \to \tilde{M}^1$.
$\tilde{M}^1$ is of the form $M[\tilde{H}^0 * H^1]$ where M is an iterated ultrapower of V generated by a normal (linear) iterated ultrapower $T = \langle M_\alpha, \pi_{\alpha,\beta} \mid \alpha \leq \beta \leq \theta\rangle$[6] and $\tilde{H}^0 * H^1 \subset j(\mathbb{P}^0 * \mathbb{P}^1)$ is generic over M. By Lemma 5, $j``G^0$ is compatible with every (reasonable) ordinal assignment to the value $j(f^{\dot{G}^0}_\kappa)(\kappa)$. Let $H^0 \subset j(\mathbb{P}^0)$ be the M generic filter obtained by swapping this generic value so that $f^{H^0}_{j(\kappa)}(\kappa) = \kappa^{++}_V$. We point out that the last modification does not change $j(\mathbb{P}^1)$ since it does not effect the coherent sequence of normal measures $j(\vec{U}^0 \restriction \kappa) = \{U^0_{\alpha,\tau} \mid \alpha < j(\kappa), \tau < o^M(\alpha)\}$ by which $j(\mathbb{P}^1)$ is defined. Therefore $H^1 \subset j(\mathbb{P}^1)$ remains generic over $M[H^0]$.
Let $j^1 : V[G^0 * G^1] \to M[H^0 * H^1]$ be the resulting modified embedding. For notational simplicity, we denote $f^{G^0}_\kappa$ by $f : \kappa \to \kappa$. We conclude that $j^1(f)(\kappa) = \kappa^{++}$ and note that $j^1(f)(\kappa^{++}) = 0$ by our choice of the measure $i^1(U^1_0)$ extending $i^1(U^0_{\kappa,0})$. It follows that $j^1(f) \restriction \kappa^{++}_V + 1 : \kappa^{++}_V + 1 \to \kappa^{++}_V + 1$.
We turn to define $\mathbb{P}^2$. $\mathbb{P}^2 = \mathbb{P}^2_{\kappa+1} = \langle \mathbb{P}^2_\alpha, \mathbb{Q}^2_\alpha \mid \alpha \leq \kappa\rangle$ is a nonstationary support iteration incorporating generalized Sacks posets, Cohen posets , Levy Collapse posets, and the coding posets described in Section 1.4. For each $\alpha < \kappa$, $\mathbb{Q}^2_\alpha$ is non-trivial if and only if it satisfies the following conditions.

1. α is inaccessible.
2. $\mathbb{Q}^1_\alpha$ is trivial.
3. $f[\alpha] \subset \alpha$ (i.e., α is a closure point of f).
4. $f(\alpha)$ is a non-measurable, inaccessible cardinal above α.

[6]Therefore, $\pi_{0,\theta} : M_0 \to M_\theta$ coincides with $j = \tilde{j}^1 \restriction V : V \to M$

5. $f[f(\alpha)+1] \subset f(\alpha)+1$.

Whenever nontrivial, $\mathbb{Q}^2_\alpha = \mathrm{Sacks}^*(\alpha, f(\alpha)) * \mathrm{Add}(\alpha^+) * \mathrm{Col}(\alpha^+ < f(\alpha)) * \mathrm{Code}(\alpha, X_\alpha)$ where $X_\alpha \subset f(\alpha)$ encodes the following generic information via a fixed simple encoding recipe.

1. The generic filter $G^0_{f(\alpha)+1} \subset \mathbb{P}^0_{f(\alpha)+1}$.
2. The generic filter $G^1_{f(\alpha)+1} \subset \mathbb{P}^1_{f(\alpha)+1}$.
3. The restriction $f \upharpoonright f(\alpha)+1 \subset (f(\alpha)+1)^2$.
4. The generic filter $G^2_\alpha \subset \mathbb{P}^2_\alpha$.
5. The generic filters $G(\mathrm{Sacks}^*(\alpha, f(\alpha)))$ and $\mathrm{Add}(\alpha^+) * \mathrm{Col}(\alpha^+, < f(\alpha))$.

$\mathbb{Q}^2_\kappa$ is defined similarly. $\mathbb{Q}^2_\kappa = \mathrm{Sacks}^*(\kappa, \kappa^{++}) * \mathrm{Add}(\kappa^+) * \mathrm{Col}(\kappa^+, < \kappa^{++}) * \mathrm{Code}(\kappa, X_\kappa)$ where $X_\kappa \subset \kappa^{++}$ encodes the following generic information.

1. The M generic filter $H^0_{\kappa_V^{++}+1} \subset j(\mathbb{P}^0)_{\kappa_V^{++}+1}$.
2. The M generic filter $H^1_{\kappa_V^{++}+1} \subset j(\mathbb{P}^1)_{\kappa_V^{++}+1}$.
3. The restriction $j(f) \upharpoonright \kappa_V^{++}+1$.
4. The V (and M) generic filter $G^2_\kappa \subset \mathbb{P}^2_\kappa$.
5. The V (and M) generic filters $G(\mathrm{Sacks}^*(\kappa, \kappa^{++}))$ and $\mathrm{Add}(\kappa^+) * \mathrm{Col}(\kappa^+, < \kappa^{++})$.

By Lemma 15, $\mathbb{P}^2$ preserves all cardinals $\beta < \kappa$ which do not belong to an interval $(\alpha^+, f(\alpha))$, where $\mathbb{Q}^2_\alpha$ is a nontrivial forcing. Let $G^2 \subset \mathbb{P}^2$ be a generic filter over $V[G^0 * G^1]$ and consider $j^1(\mathbb{P}^2)$. Our choice of $\mathbb{Q}^2_\kappa$ implies $G^2 \subset j^1(\mathbb{P}^2) \upharpoonright \kappa+1$ is generic over $M[G^0 * G^1]$. Moreover, the forcing $j(\mathbb{P}^2)/(G^2) \upharpoonright (\kappa, \kappa_V^{++}]$ is trivial, and the pointwise image j^1"G^2 of G^2 generates a generic filter $H^2 \subset j^1(\mathbb{P}_2)/G^2$ over $M[H^0 * H^1 * G^2]$. Let $j^2 : V[G^0 * G^1 * G^2] \to M[H^0 * H^1 * H^2]$ denote the resulting extension of j^1 and set $G = G^0 * G^1 * G^2$, $H = H^0 * H^1 * H^2$, and $U^2 = \{X \subset \kappa \mid \kappa \in j^2(X)\}$. Note that $j^2 : V[G] \to M[H]$ coincides with the ultrapower embedding of $V[G]$ by U^2. Theorem 1 is an immediate consequence of the following result.

Proposition 17. *U^2 is the only normal measure on κ in $V[G]$.*

Proof. Let U be a normal measure on κ, $i_U : V \to N_U \cong \mathrm{Ult}(V[G], U)$. Then $N_U = N[I]$ where N is an iterated ultrapower of V generated by a normal iteration $T^N = \langle N_\alpha, \pi^N_{\alpha,\beta} \mid \alpha \le \beta \le \theta^N \rangle$, for which $i = i_U \upharpoonright V$ coincides with π^N_{0,θ^N}, and $I = I^0 * I^1 * I^2 \subset i(\mathbb{P}^0 * \mathbb{P}^1 * \mathbb{P}^2)$ is generic over N with i"$G \subset I$.

Our goal is to show that $U = U^2$. The argument is based on an analysis of the iteration T^N and the generic filter I. We first introduce several iteration related notations, which will be frequently used throughout the proof.

Definition 18 (Associated generators and key generators)**.** We say that $\delta < i(\kappa)$ is a **generator** of the iteration T^N if it is a critical point of one of the measure ultrapowers of T^N. Let $\gamma \le i(\kappa)$ be a measurable cardinal in N. γ is represented in some finite iterated ultrapower by a finite subiteration of T^N. Let $\pi^\gamma : V \to N^\gamma$ be a minimal finite iterated ultrapower which contains a representation γ' of γ. This means there is a unique complementary iterated ultrapower $T^{N^\gamma, N}$ starting from N^γ and resulting in a direct limit embedding $k^\gamma : N^\gamma \to N$ so that $i = k^\gamma \circ \pi^\gamma$. We define the **generators of T^N associated with** γ to be γ' and all its images under the iteration maps of complementary iteration $T^{N^\gamma, N}$.
We define the **Key generators** of the iteration T^N to be the generators associated with $i(\kappa)$. These are κ and all its images under the iteration maps in T^N.

For the rest of the argument we denote κ_V^{++} by η. We separate the argument showing $U = U^2$ into four claims.

Claim 1: The iterations T^N and T agree on all critical points and measures up to η.
To see this, note that $\mathcal{P}(\kappa)^{N_U} = \mathcal{P}(\kappa)^{V[G]}$. It follows that $i(\kappa) > (2^\kappa)^{N_U} = (2^\kappa)^{V[G]} \geq \eta$, and that stage κ of $i_U(\mathbb{P}^2)$ is nontrivial [7]. Next, N and V agree on all definable subset of $H^{\mathcal{K}}_{\kappa^+}$ and in particular, on the sequence $\vec{S}_\kappa$ upon which $\mathrm{Code}_0(\kappa)$ is defined. Therefore H^2 and I^2 must agree on the coding generic closed unbounded subsets added at stage κ and thus, also on coded information which includes the $i(\mathbb{P}^1)$ ($j(\mathbb{P}^1)$ respectively) generic Prikry/Magidor/Radin sequences added on all measurable cardinals in N (M) up to $\eta + 1$. This implies that the Mitchell order functions o^M and o^N agree up to $\eta + 1$, which in turn, implies $T \restriction \eta = T^N \restriction \eta$. In particular, $\pi^N_{0,\eta}(\kappa) = \eta$. The agreement on the coding generic also implies that $i_U(f)(\kappa) = j^2(f)(\kappa) = \eta$. $\square$[Claim 1]

Claim 2: Both iterations T^N and T do not contain key generators above η.
We prove the result for T^N only, as the argument for T is similar. Suppose otherwise. Let $\gamma < i(\kappa)$ be a key generators above η and $\phi \in \kappa^\kappa$ in $V[G]$ such that $\gamma = i_U(\phi)(\kappa)$. The results of [5] and [3] apply to $\mathbb{P}^2$ and $\mathbb{P}^1$ respectively and guarantee that in $V[G^0]$, there is $\phi^0 : \kappa \to \kappa$ such that $\phi(\nu) \leq \phi^0(\nu)$ for all $\nu < \kappa$. Back in V, let $\dot{\phi}^0$ be a name for ϕ^0. By Lemmata 4 and 2, there is $F : \kappa \to [\kappa]^{<\kappa}$ in V, satisfying $|F(\nu)| \leq \nu^+$, a closed unbounded set $C \subset \kappa$, and $p \in G^0$, such that for all $\nu \in C$, p forces that if $\dot{f}[\nu+1] \subset (\nu+1)$ then $\phi^0[\nu] \subset F(\nu)$. By Claim 1, η is a key generator of T^N and $i_U(f) \restriction \eta+1 = j^2(f) \restriction \eta+1 : \eta+1 \to \eta+1$. Hence, $i_U(p) \in I^0$ forces $i_U(\phi)[\eta] \subset i(F)(\eta)$. In particular, $\gamma = i_U(\phi)(\kappa) \in i(F)(\eta)$. It follows that γ cannot be a key generator as $\eta < \gamma$ and $\gamma \in k^\eta(\pi^\eta(F)(\eta'))$, where the last set is of size $|\pi^\eta(F)(\eta')| < \gamma$. $\square$[Claim 2]

We have established so far that $T \restriction \eta = T^N \restriction \eta$, implying that $N_\eta = M_\eta$ and $\pi_{0,\eta}(\kappa) = \pi^N_{0,\eta}(\kappa) = \eta$. As $\eta < i(\kappa), j(\kappa)$, both $T \setminus \eta, T^N \setminus \eta$ must include ultrapowers by measures on η. We claim that these measures must both be the zero measure $U^{N_\eta}_{\eta,0} = U^{M_\eta}_{\eta,0}$. To see this, note that we showed κ is a nontrivial forcing stage of both $i_U(\mathbb{P}^2)$ and $j^2(\mathbb{P}^2)$. By the definition of $\mathbb{P}^2$, $\eta = i_U(f)(\kappa) = j^2(f)(\kappa)$ is inaccessible but not a measurable. Therefore $\pi_{\eta,\eta+1} = \pi^N_{\eta,\eta+1}$ is an ultrapower of $N_\eta = M_\eta$ by the same measure, $U_{\eta,0}$.

To show $T = T^N$, it remains to verify $T \setminus \eta + 1 = T^N \setminus \eta + 1$. By standard arguments, both iterations T, T^N cannot iterate a specific measure more than ω many times consecutively. Since by Claim 2, $T \setminus \eta, T^N \setminus \eta$ do not contain key generators, it follows that $i(\kappa) = \pi^{N_{\eta+1}}(\kappa) = \pi^{M_{\eta+1}}(\kappa) = j(\kappa)$ and that all ultrapowers of $T^N \setminus \eta+1$ and $T \setminus \eta+1$ and taken by measures on ordinals in $(\eta, j(\kappa)) = (\eta, i(\kappa))$.

Claim 3: $\mathcal{T} \setminus \eta + 1 = \mathcal{T}^N \setminus \eta + 1$.
The proof relies on an analyze the Prikry/Magdor/Radin generic cofinal sequence determined by the generic filter $I^1 \setminus (\eta+1)$. The idea is that the generators of the iteration T, T^N above η conincide with tail segments of the Prikry/Magidor/Radin generic sequences of $I^1 \setminus (\eta+1)$. In this sense, one can view the iterated ultrapower of $\mathcal{T}^N \setminus \eta+1$ can be seen as the minimal iterated ultrapower of $N_{\eta+1}$ needed to generate Prikry/Magidor/Radin generic sequences, making the iteration definable from $N_{\eta+1}$. Since $N_{\eta+1} = M_{\eta+1}$. The same analysis, applied to $H^1 \setminus (\eta+1)$, shows that $\mathcal{T} = \mathcal{T}^N$.
For every N measurable cardinal $\gamma \in (\eta, j(\kappa))$, let $c^\gamma = c^{I^1}_\gamma$ denote the generic Prikry/Magidor/Radin sequence assigned to γ by the generic $I^1 = i_U(G^1)$. The main technical result of our analysis asserts that for every measurable cardinal $\gamma > \eta$ in N, c_γ consists of an initial segment t determined by $i_U(p)$ for some $p \in G^1$ and an end segments $c_\gamma \setminus (\max(t))$ which coincides with the set of all generators of T^N associated with γ. To this end, note that as there are no key generators above η, then the statement of Lemma 2 applies to all $\leq^*$ dense open subsets of

[7]This is the only stage allowing to add η subsets of κ.

$i_U(\mathbb{P}^1) \setminus \eta$. Therefore, i_U"G^1 meets every $\leq^*$ dense open set of $i_U(\mathbb{P}^1) \setminus \eta$. Fix $\gamma \in (\eta, i(\kappa))$, a measurable cardinal in N, and let $\pi^\gamma : V \to N^\gamma$ be a minimal finite subiteration of $\mathcal{T}^N$ which represents γ by some $\gamma' \in N^\gamma$. The minimality of N^γ implies that γ' is not a key generator of N^γ. Hence, $\gamma' = \pi^\gamma(h)(\nu_0, \ldots, \nu_k)$, where $\nu_0, \ldots, \nu_k < \gamma'$ are generators of N^γ. The fact $\mathbb{P}^1$ is a nonstationary support iteration implies there exists a name of some $p \in G^1$ so that γ' is forced to be in $\mathrm{dom}(\pi^\gamma(p))$. The support restriction of $\mathbb{P}^0$ implies that if t' be the $\pi^\gamma(\mathbb{P}^1)$ name for the stem of $\pi^\gamma(p)(\gamma')$ then t' is the stem of $\pi^\gamma(q)(\gamma')$ for every $q \geq p$ in G^1. Let $\mathcal{T}^{N^\gamma, N}$ be the complementary iterated ultrapower from N^γ to N and denote the resulting embedding by $k^\gamma : N^\gamma \to N$. It is easy to see that for an ordinal τ to appear in a stem of a condition $\langle t^*, T^* \rangle$ extending $i_U(p)(\gamma) = k^\gamma(\pi^\gamma(p)(\gamma'))$ and compatible with i_U"G^1, it is necessary that τ appears in $k^\gamma(T')$, for all trees T' such that $\pi^\gamma(q)(\gamma') = \langle t', T' \rangle$ for some $q \in G^1$. Note that the set of all ordinals τ satisfying the last restriction is the set of all generators associated with γ. Thus, $c_\gamma \setminus t'$ is contained in the set of $\mathcal{T}^N$ generators associated with γ.
We claim $c_\gamma \setminus t'$ contains all $\mathcal{T}^N$ generators $\delta < \gamma$ associated with γ. Let us first point out that it is actually sufficient to verify that every generator δ of T^N appears in c_γ for some γ associated with δ. To see this suffices, consider the structure of Prikry/Magidor/Radin generic sequences. If $o(\delta) = \rho$ then $o(\gamma) > \rho$ and there is another key generator $\tau \in c_\gamma$ associated with γ such that $o(\tau) = \rho + 1$ and $\gamma \in c_\tau$. However, there is only one generator $\tau < i(\kappa)$ so that $o(\tau) = \rho + 1$ for which δ is associated with[8]. Therefore, the fact every generator δ of T^N appears in some c_γ completely characterizes c_τ where $o(\tau)$ is a successor ordinal. It is then straightforward to verify by induction on $\gamma < i(\kappa)$ the same description applies to all c_γ.
We proceed to verify $\delta \in c_\gamma$ for some γ associated with δ. Pick $h \in {}^\kappa\kappa \cap V[G]$ so that $\delta = i_U(h)(\kappa)$. By Lemma 15, there exists $h' : \kappa \to \mathcal{P}_\kappa(\kappa)$ in $V[G^0 * G^1]$ satisfying $|h'(\alpha)| \leq f(\alpha)$ and $h(\alpha) \in h'(\alpha)$ for all $\alpha < \kappa$.

For each α, let $\{\mu_\alpha(i) \mid i < f(\alpha)\}$ be an enumeration of $h'(\alpha)$. For every condition $p \in \mathbb{P}^1$, standard arguments concerning dense open subsets of Prikry type forcings[9] asserts there exists a direct extension p^* of p and an assignment $(\alpha, i) \mapsto \vec{T}(\alpha, i)$ for each $\alpha \in \bar{C}$ and $i < f(\alpha)$, such that $\vec{T}(\alpha, i)$ is a finite sequence of fat trees on cardinals above $f(\alpha)$, such that for every sequence of maximal branches $\vec{\sigma}$ through the trees of $\vec{T}(\alpha, i)$, the extension of $p*$ by $\vec{\sigma}$ decides the value of $\mu_\alpha(i)$. Let $\{\mu_\kappa(i) \mid i < \eta\} = i_U(h')(\kappa)$. Suppose $i^* < \eta$ is the value for which $\delta = i_U(h')(i^*)$. Let $\pi^\delta : V \to N^\delta$ be a finite iterated ultrapower where this information about δ, η, i^* is represented by some δ', η', i' respectively. By elementarity, $\pi^\delta(p^*)$ forces there is a finite sequence of fat trees deciding the value of $\mu_\gamma(i')$. Let $k^\delta : N^\delta \to N$ be the complementary elementary embedding. It follows that in $N[I^0]$ there is a sequence of maximal branches $\vec{\sigma} = \langle \sigma_0, \ldots, \sigma_k \rangle$ through $T_0, T_1, \ldots, T_k$ respectively, consisting of generators which force $\delta = \mu_\kappa(i^*)$.

Sub Claim 3.1: We may assume all ordinals in $\bigcup_{1 \leq l \leq k} \sigma_l$ are at most δ.
By this, we mean that if one of the elements $\tau \in \sigma_l$ is above δ then we may reduce the fat tree S_i associated with τ to a smaller tree which still decides $\delta = \mu_\kappa(i^*)$. To see this, suppose that τ is the largest ordinal in σ_k and $\tau > \delta$. Let γ be the measurable cardinals associated with the fat tree S^k from which σ^k is taken. Let N' be a finite iterated ultrapower which embeds into N in which $\gamma, S^k, \sigma_k \setminus \{\tau\}, \delta$, and σ_l, $l < k$ are represented by $\gamma, S', \sigma', \delta'$, and σ'_l respectively. Let $U^{N'}_{\gamma', \rho'}(s')$ denote measure associated with the splitting set $\mathrm{succ}_{S'}(\sigma')$ of the fat tree S'[10]. Consider the projection map π of $U^{N'}_{\gamma', \rho'}(s')$ to the normal measures $U^{N'}_{\gamma', 0}(\emptyset)$ defined

[8] I.e., τ is the unique ordinal or order $o(\tau) = \rho + 1$ so that if N_τ is a minimal finite ultrapower containing representation τ' of τ, then there exists some $n < \omega$ so that when iterating the ρ-th measure on τ' n times, then one of the critical points represents γ.

[9] e.g., see [10].

[10] This is a $i(\mathbb{P}^0)$ extension of the measure $U^{N'}_{\gamma', \rho'}$ concentrating on the set of measurable $\alpha < \gamma'$ for which $c_{s'} := \bigcup_{\nu \in s'} c_\nu$ is an initial segment of c_α. See [7] for more information.

by $\pi(\nu) = \min(c_\nu \setminus \max(s')+1)$. Let $\nu \mapsto \delta_\nu$ be the map defined on $\mathrm{succ}_{S'}(\sigma')$ where for each ν, δ_ν is the ordinal forced to be $\mu_{\gamma'}(i')$. Define $A' = \{\nu \in \mathrm{succ}_{S'}(\sigma') \mid \delta'_\nu < \pi(\nu)\}$. We claim $A' \in U^{N'}_{\gamma',\rho'}(s')$. Suppose otherwise. For each $\epsilon < \gamma'$ the set $\{\nu \mid \delta_\nu = \epsilon\}$ does not belong to $U^{N'}_{\gamma',\rho'}(s')$. In particular $\{\nu \mid \delta_\nu = \delta'\} \notin U^{N'}_{\gamma',\rho'}(s')$ contradicting the fact $\delta_\tau = \delta$ in N. Therefore $A' \in U^{N'}_{\gamma',\rho'}(s')$ and by [8], we can press down on the map $\nu \mapsto \delta_\nu$ and fix the value on a measure one set. Since $\delta_\tau = \delta$ in N the constant fixed value in N' must be δ'. We conclude that the top level of the tree S' (S^k respectively) is redundant for validating the assertion $\mu_\gamma(i') = \check{\delta}$. $\square$[Sub Claim 3.1]

Sub Claim 3.2: δ is the maximal ordinal of σ_k. Suppose otherwise. δ cannot be expressed as $i(\phi)(\sigma_0, \ldots, \sigma_k)$ for some function $\phi \in V$ as it is a generator of the iteration T^N. However, as we force with $\mathbb{P}^1$ over $V[G^0]$ and not over V, there might be a function $\phi \in V[G^0]$ with this property. To see this is impossible, consider a $\mathbb{P}^0$ name $\dot{\phi}$ of ϕ. Let $\pi' : V \to N'$ be a finite iterated ultrapower which represents η and σ_l for each $l \leq k$. Let $k' : N' \to N$ be the complementary direct limit embedding and denote $(k')^{-1}(\eta)$ and each $(k')^{-1}(\sigma_l)$ by $\eta', \sigma'_l \in N'$ respectively. By Lemma 4 there are $q \in G^0$, $C \subset \kappa$ closed unbounded, and $F : \kappa \to \mathcal{P}(\kappa)$ satisfying $|F(\alpha)| \leq h(\alpha)$ such that

$$q \Vdash \left(\forall \alpha \in \check{C}. \dot{f}[\alpha+1] \subset \alpha+1 \longrightarrow \dot{\phi}[h(\alpha)^k] \in F(\alpha)\right).$$

Let $X' = \pi'(F_{\dot{\phi}})(\eta')$. On one hand, the fact $N' \models |X'| = i(h)(\alpha)$ implies $\delta \notin X = k'(X')$. On the other hand, the fact $\eta \in i(C)$ and $i_U(f)[\eta+1] \subset \eta+1$ implies $\delta = \phi(\sigma_0, \ldots, \sigma_k) \in i(F_{\dot{\phi}})(\eta) = X$. Contradiction. $\square$[Sub Claim 3.2]

Let us explain how the above results determine the generic sequences $c_\gamma = c^{I^1}_\gamma$ for every measurable $\gamma \in (\eta, i(\kappa))$ and by this, the iteration $T^N \setminus \eta+1$. Let $e^* = \pi^N_{\eta,\eta+1} : N_\eta \to N^* \cong \mathrm{Ult}(N_\eta, U^{N_\eta}_{\eta,0})$ denote the ultrapower of N_η by the zero measure $U^{N_\eta}_{\eta,0}$ and $i^* = e^* \circ \pi_{0,\eta}$. All measurable cardinals $\gamma \in (\eta, i(\kappa))$ are represented in a finite iterated ultrapower of N^*. Let $\gamma \in (\eta, i(\kappa))$. Then γ is represented by some γ' in a finite iterated ultrapower of N'. Suppose that $\gamma' \in N^*$. Then $\gamma' > \eta$ has no associated generators in N^*. Let t' be the $i(\mathbb{P}^0)$ name for a stem of the c_γ determined by i_U"G^1. Since $c_\gamma \setminus \max(t')$ is the set of all generators δ associated with γ, it follows that the iteration from N^* to N iterates all the measures on γ' (i.e., according to its Mitchell order) precisely the number of times needed to obtain the appropriate Prikry/Magidor/Radin cofinal sequence of the right ordertype. Note that we cannot iterate more than necessary, since this would create a generator δ of the iteration T^N which does not appear on c_τ for any associated generator τ. The same description applies to measurable cardinals γ represented in a finite iterated ultrapower of N^*.
Finally, as mentioned at the beginning of the proof of Claim 3, the same analysis applies to the iteration $T \setminus \eta+1$. Since the description of the iteration $T \setminus \eta+1$ relies on $M_\eta = N_\eta$ and $U^{M_\eta}_{\eta,0} = U^{N_\eta}_{\eta,0}$ it follows that $T \setminus \eta+1 = T^N \setminus \eta+1$. $\square$[Claim 3]

Reviewing the description of the generic sequence $c_\gamma = c^{I^1}_\gamma$ given in Claim 3, we see that a tail segment of each c_γ is completely determined by the generators of $\mathcal{T}^N \setminus \eta+1 = \mathcal{T} \setminus \eta+1$, and therefore must agree with the U^2 ultrapower sequence $c^{H^1}_\gamma$ on this tail segment. To show $I^1 = H^1$, we need to verify that for each measurable $\gamma \in (\eta, i(\kappa))$, $c^{I_1}_\gamma$ and $c^{H^1}_\gamma$ agree on the initial segment determined by i_U"G^1 (j^2"G^1 respectively). The embeddings $i = i_U \restriction V$ and $j = j^2 \restriction V$ result from the same iteration $\mathcal{T}^N = \mathcal{T}$ and thus must be equal. Note however that i_U"G^1 (j^2"G^1 respectively) relies on information given by the extension $i^0 : V[G^0] \to N[I^0]$ ($j^0 : V[G^0] \to N[H^0]$ respectively) of i. Therefore, at this point, showing $I^1 = H^1$ amounts to verifying $H^0 = I^0$.

Claim 4: $H^0 * H^1 * H^2 = I^0 * I^1 * I^2$.

The choice of the coding poset at stage κ guarantees that H^0, I^0 and H^1, I^1 agree everywhere up to η, and that $H^2 \restriction (\kappa+1) = I^2 \restriction (\kappa+1)$. We verify H and I agree everywhere else. Let us first verify $H^0 = I^0$. We know $H^0 \restriction \eta + 1 = I^0 \restriction \eta + 1$. Since there are no key generators between η and $i(\kappa)$, the proof of Lemma 4 guarantees $j``G^0 \restriction \kappa = i``G^0 \restriction \kappa$ completely determines $H^0 \restriction (\eta, i(\kappa)) = I^0 \restriction (\eta, i(\kappa))$. Similarly, the argument of Lemma 5 implies $i``G^0(\kappa) = j``G^0(\kappa)$ determines the values of the functions $i(\dot{f_\kappa}) \setminus (\eta, i(\kappa))$, $j(\dot{f_\kappa}) \setminus (\eta, i(\kappa))$ at all ordinals δ which are not generators of $T = T^N$. Let δ be a generator of $T \setminus \eta$. By Sub Claim 3.2, δ belongs to generic Prikry/Magidor/Radin sequence c_γ. Since the measures $U^0_{\alpha,\tau} \in \vec{U}^0 \restriction \kappa$ used for the definition of $\mathbb{P}^1$ concentrate on the set $\{\delta < \alpha \mid f_\alpha(\delta) = 0\}$ it follows that $i_U(f_\kappa)_\gamma(\delta) = 0 = j^2(f_\kappa)(\delta)$ for each generator $\delta \in (\eta, i(\kappa))$. This shows $I^0 = H^0$, and by the description following the proof of Claim 3, we conclude $H^1 = I^1$. Finally, we verify $H^2 = I^2$. Lemma 4 guarantees $H^2 \restriction i(\kappa) = I^2 \restriction i(\kappa)$. Furthermore, the proof of Lemma 16 guarantees $i``G^2(\kappa) = j``G^2(\kappa)$ completely determines a $i(\mathbb{Q}_\kappa)$ generic filter.[11] Hence $H^2(\kappa) = I^2(\kappa)$. $\square$[Claim 4]

$\square$

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