Math 33b **Midterm 1**
Lecture 2
Fall 2007

Name: ____________________

Section: ____________________

Instructions:

- There are 6 problems. Make sure you are not missing any pages.
- Show all work in detail or your answer will not receive credit.
- Write neatly and box all answers.
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.
- Do not use your own scratch paper. Extra scratch paper is available from the front of the room.

Question	Points	Score
1	20	
2	20	
3	20	
4	20	
5	10	
6	10	
Total:	100	

1. (20 points) Solve the initial value problem:

$$t\frac{dy}{dt} + (2t^2 + 3t^3)y = te^{-(t^2+t^3)}, \quad y(0) = 5$$

We rewrite it:

$$\frac{dy}{dt} + (2t + 3t^2)y = e^{-(t^2+t^3)}$$

or

$$(e^{t^2+t^3}y)' = e^{t^2+t^3}\frac{dy}{dt} + (2t + 3t^2)e^{t^2+t^3}y = 1.$$

So

$$e^{t^2+t^3}y = \int 1 \, dt = t + C,$$

giving

$$y = e^{-t^2-t^3}t + Ce^{-t^2-t^3}.$$

Finally

$$5 = y(0) = C$$

so

$$y = e^{-t^2-t^3}t + 5e^{-t^2-t^3}.$$

2. (a) (10 points) Solve the initial value problem:

$$\frac{dy}{dt} = 2t(y-1)^2, \quad y(0) = 0$$

Rewrite it:

$$\frac{\frac{dy}{dt}}{(y-1)^2} = 2t$$

Integrating both sides with respect to t

$$-(y-1)^{-1} = t^2 + C$$

So

$$y = 1 - \frac{1}{t^2 + C}.$$

Finally

$$0 = y(0) = 1 - \frac{1}{C}$$

so

$$y = 1 - \frac{1}{t^2 + 1}.$$

(b) (10 points) Solve the initial value problem:

$$\frac{dy}{dt} = 2t(y-1)^2, \quad y(0) = 1$$

Since $2t(1-1)^2 = 0$ for every t, we have the constant solution

$$y(t) = 1$$

which satisfies the initial conditions.

3. (a) (10 points) Show that the following equation is exact:

$$2x + y^2 + 2xy\frac{dy}{dx} = 0$$

Well,

$$\frac{\partial}{\partial x}2xy = 2y = \frac{\partial}{\partial y}(2x + y^2)$$

for every x, y, so by the theorem in the book

$$2x + y^2 + 2xy\frac{dy}{dx} = 0$$

must be exact.

(b) (10 points) Find the explicit solution to the initial value problem:

$$2x + y^2 + 2xy\frac{dy}{dx} = 0, \quad y(1) = 1$$

We know that the general solution is given implicitly by $F(x, y) = C$ where $\frac{\partial}{\partial y}F = 2xy$ and $\frac{\partial}{\partial x}F = 2x + y^2$. So

$$F(x, y) = \int 2xy\ dy + u(x) = xy^2 + u(x).$$

and

$$2x + y^2 = \frac{\partial}{\partial x}F = \frac{\partial}{\partial x}(xy^2 + u(x)) = y^2 + u'(x).$$

So we need $u'(x) = 2x$; in particular $u = x^2$ works fine. So, the general solution is

$$xy^2 + x^2 = C$$

or

$$y = \frac{+}{-}\sqrt{\frac{C}{x} - x}.$$

Finally

$$1 = y(1) = \frac{+}{-}\sqrt{C - 1}$$

So, we need $+$ and $C = 2$, giving the solution

$$y = \sqrt{\frac{2}{x} - x}.$$

4. (20 points) Find an integrating factor for the following equation (you do not need to solve the equation):

$$4x^2y + 2y^2 + (3x^3 + 4xy)\frac{dy}{dx} = 0$$

Hint: It is of the form $\mu(x, y) = xy^a$ for some a.

What we need is

$$xy^a(4x^2y + 2y^2) + xy^a(3x^3 + 4xy)\frac{dy}{dx} = 0$$

to be exact. By the theorem in the book, this will happen if

$$4(a+1)x^3y^a + 2(2+a)xy^{1+a} = \frac{\partial}{\partial y}xy^a(4x^2y+2y^2) = \frac{\partial}{\partial x}xy^a(3x^3+4xy) = 12x^3y^a + 8xy^{1+a}$$

for every x, y. i.e., we need $4(a+1) = 12$ and $2(2+a) = 8$. So, $a = 2$ works, giving the integrating factor

$$\mu(x, y) = xy^2.$$

5. (10 points) Solve the initial value problem:

$$\frac{dy}{dx} = \sin(\pi y)(2^y - 4)(y^3 - 6y^2 + 12y - 8), \quad y(1) = 2$$

Well, definitely $\sin(\pi \cdot 2)(2^2 - 4)(2^3 - 6 \cdot 2^2 + 12 \cdot 2 - 8) = 0$ so $y(x) = 2$ is a solution and satisfies the initial conditions.

6. (10 points) Find the equilibrium solutions for the following equation, and identify them as asymptotically stable or unstable.

$$\frac{dy}{dt} = y^2 - 5y + 6$$

$$y^2 - 5y + 6 = (y-2)(y-3)$$

so the equilibrium solutions are

$$y(t) = 2, y(t) = 3.$$

Since $(y-2)(y-3) < 0$ for $y = 2 + \epsilon$ and $(y-2)(y-3) > 0$ for $y = 2 - \epsilon$, we see that $y(t) = 2$ is asymptotically stable.

Since $(y-2)(y-3) > 0$ for $y = 3 + \epsilon$ and $(y-2)(y-3) < 0$ for $y = 3 - \epsilon$, we see that $y(t) = 3$ is asymptotically unstable.

Extra Scratch Paper: