Math 131a
Lecture 2
Spring 2009

Practice Midterm 2

Name: ______________________________

Instructions:

- There are 4 problems. Make sure you are not missing any pages.
- Unless stated otherwise (or unless it trivializes the problem), you may use without proof anything proven in the sections of the book covered by this test (excluding the exercises).
- Give complete, convincing, and clear answers (or points will be deducted).
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Suppose that $a \leq b$ and $(x_n)_{n=1}^{\infty}$ is a convergent sequence with $x_n \in [a, b]$ for every n (here $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$). Prove that $\lim_{n\to\infty} x_n \in [a, b]$.

Solution: Suppose not, and that $c = \lim_{n\to\infty} x_n$. Since $c \notin [a, b]$, we either have $c > b$ or $c < a$. We will consider former case, the latter is completely analogous. Since $c > b$, we have $c - b > 0$, so (taking $\epsilon = c - b$ in the definition of the limit) there is an $N \in \mathbb{N}$ such that $n \geq N$ implies $|x_n - c| < c - b$. In particular, if we have $c - x_N < c - b$ and so $x_N > b$ contradicting the fact that $x_N \in [a, b]$.

2. (10 points) Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence satisfying $|x_n - x_{n+1}| \leq 2^{-n}$ for every n. Prove that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution:

Let $\epsilon > 0$. We need to show that there is an N such that for all $n, m \geq N$ we have $|x_n - x_m| < \epsilon$. Assume $m < n$. Then

$$x_n - x_m = \sum_{j=m}^{n-1} x_{j+1} - x_j.$$

So, by repeated applications of the the triangle inequality

$$|x_n - x_m| \leq \sum_{j=m}^{n-1} |x_{j+1} - x_j| \leq \sum_{j=m}^{n-1} 2^{-j}$$

The side above is

$$= \sum_{j=0}^{n-m-1} 2^{-(j+m)} = 2^{-m} \sum_{j=0}^{n-m-1} 2^{-j} \leq 2^{-m} \sum_{j=0}^{\infty} 2^{-j} = 2^{-m+1}.$$

Above, the last equation is from the formula for the sum of a geometric series. It follows that if we choose N large enough so that $2^{-N} < \epsilon/2$ we have $|x_n - x_m| < 2 \cdot 2^{-\min(m,n)} \leq 2 \cdot 2^{-N} < \epsilon$ for all $m, n \geq N$.

3. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined $f(x) = 0$ if x is irrational and $f(x) = x$ if x is rational. Show that f is continuous at 0.

Solution:

Since $f(0) = 0$, we need to show that $\lim_{x\to 0} f(x) = 0$. Let $\epsilon > 0$. (Taking $\delta = \epsilon$) Suppose $0 < |x - 0| < \epsilon$. Either x is irrational or rational. In the former case, we have $|f(x) - 0| = 0 < \epsilon$ as desired. In the latter case, we have $|f(x) - 0| = |x - 0| < \epsilon$, as desired.

4. (10 points) Let $(x_n)_{n=1}^{\infty}$ be a convergent sequence. Prove (using only the definitions) that $(x_n)_{n=1}^{\infty}$ is a Cauchy sequence.

Solution:

Let $\epsilon > 0$. We need to find an N such that for all $m, n \geq N$ we have $|x_n - x_m| < \epsilon$. Since (x_n) is convergent, it has some limit L and there is an N so that for all $j \geq N$, $|x_j - L| < \epsilon/2$. Then, for $m, n \geq N$ we have $|x_m - x_n| = |x_m - L + L - x_n| \leq |x_m - L| + |x_n - L| < \epsilon/2 + \epsilon/2 = \epsilon$ as desired.

Extra Scratch Paper: