Math 131a
Lecture 2
Spring 2009

Midterm 2

Name: ____________________

Instructions:

- There are 4 problems. Make sure you are not missing any pages.
- Unless stated otherwise (or unless it trivializes the problem), you may use without proof anything proven in the sections of the book covered by this test (excluding the exercises).
- Give complete, convincing, and clear answers (or points will be deducted).
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Let $(x_n)_{n=1}^{\infty}$ be a bounded monotonically increasing sequence. Prove that $\sup\{x_n : n \in \mathbb{N}\} = \lim_{n\to\infty} x_n$.

Solution:

Denote $\alpha = \sup\{x_n : n \in \mathbb{N}\}$. Then $\alpha \in \mathbb{R}$ since $(x_n)_{n=1}^{\infty}$ is bounded. Let $\epsilon > 0$. We need to find an N so that for all $n \geq N$, we have $|x_n - \alpha| < \epsilon$. Since $\alpha - \epsilon < \alpha$ and α is the *least* upper bound, there must be a $y \in \{x_n : n \in \mathbb{N}\}$ such that $y > \alpha - \epsilon$. Since $y \in \{x_n : n \in \mathbb{N}\}$ there is an $N \in \mathbb{N}$ such that $y = x_N$. For $n \geq N$ we then have $y - \epsilon < x_N \leq x_n \leq y$ (here the second inequality follows from the fact that $(x_n)_{n=1}^{\infty}$ is monotonically increasing, and the last inequality follows from the fact that y is an upper bound for $\{x_n : n \in \mathbb{N}\}$) and in particular, $|x_n - y| < \epsilon$.

2. (10 points) Let $(x_n)_{n=1}^{\infty}$ be a real-valued sequence satisfying $|x_{n+1} - x_n| < \frac{1}{n}$ for every n. Is it necessarily true that $(x_n)_{n=1}^{\infty}$ is a Cauchy-sequence? Prove your answer.

Solution:

We do not necessarily have $(x_n)_{n=1}^{\infty}$ Cauchy. Indeed consider the sequence $x_n = \sum_{j=1}^{n} \frac{1}{j}$. Then $|x_{n+1} - x_n| = \frac{1}{n+1} < 1/n$. However, we know (say, by the integral test since $1/x$ is decreasing and nonnegative and $\lim_{n\to\infty} \int_1^n \frac{1}{x}\, dx = \infty$) that

$$\lim_{n\to\infty} x_n = \sum_{j=1}^{\infty} \frac{1}{j} = \infty$$

and in particular $(x_n)_{n=1}^{\infty}$ is unbounded and so it can't be Cauchy.

3. (10 points) Consider $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$. Is this sum convergent? If so, what is its limit? Prove your answer/s.

Solution:

We need to see if the sum is convergent, i.e. whether $\lim_{m\to\infty} \sum_{n=1}^{m} \frac{1}{n} - \frac{1}{n+1}$ is convergent. However, for each m we have

$$\sum_{n=1}^{m} \frac{1}{n} - \frac{1}{n+1} = \sum_{n=1}^{m} \frac{1}{n} - \sum_{n=1}^{m} \frac{1}{n+1} = \sum_{n=1}^{m} \frac{1}{n} - \sum_{n=2}^{m+1} \frac{1}{n} = 1 - \frac{1}{m+1}.$$

Clearly $\lim_{m\to\infty} 1 - \frac{1}{m+1} = 1$ (take $N \geq \frac{1}{\epsilon}$) and so $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ converges to 1.

Alternatively, to see that the sum is convergent use the comparison test. Since $|\frac{1}{n} - \frac{1}{n+1}| = \frac{1}{n(n+1)} \leq \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent (say, by the integral test since $1/x^2$ is decreasing, positive, and $\lim_{m\to\infty} \int_1^m \frac{1}{x^2}\, dx$ converges) the comparison test tells us that $\sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+1}$ is convergent (however it doesn't give us the limit).

4. (10 points) Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined $f(x) = \sin(1/x)$ if $x \neq 0$ and $f(x) = 0$ if $x = 0$. Is $\lim_{x\to 0} f(x)$ convergent? (i.e. is there an $L \in \mathbb{R}$ such that $L = \lim_{x\to 0} f(x)$?). If so, what is the limit? Prove your answer/s.

Solution:

No, it's not convergent. To see this, we will suppose that it is convergent and derive a contradiction. Suppose $L = \lim_{x\to 0} f(x)$. Then, taking $\epsilon = 1$ in the definition of limit, we see that there is a $\delta > 0$ such that for every $x \in \mathbb{R}$ with $0 < |x| < \delta$ we have $|\sin(1/x) - L| < 1$. In particular, choosing $N \in \mathbb{N}$ with $N > \frac{1}{2\pi\delta}$ and letting $x_1 = \frac{1}{\frac{\pi}{2}+2N\pi}$ and $x_2 = \frac{1}{\frac{3\pi}{2}+2N\pi}$ we have $0 < |x_1|, |x_2| < \frac{1}{2N\pi} < \delta$. Thus

$$|1 - L| = |\sin(1/x_1) - L| < 1$$

and

$$|-1 - L| = |\sin(1/x_2) - L| < 1.$$

So, by the triangle inequality, $2 = |1 - (-1)| \leq |1 - L| + |L - (-1)| < 1 + 1 = 2$, a contradiction. Thus $\lim_{x\to 0} f(x)$ is not convergent.

Extra Scratch Paper: