Math 131a
Lecture 2
Spring 2009

Midterm 1

Name: ____

Instructions:

- There are 4 problems. Make sure you are not missing any pages.
- Unless stated otherwise (or unless it trivializes the problem), you may use without proof anything proven in the sections of the book covered by this test (excluding the exercises).
- Give complete, convincing, and clear answers (or points will be deducted).
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Suppose that $f : C \to D$ is an injective function, and that $A, B \subset C$. Prove that $f(A \cap B) = f(A) \cap f(B)$. (Here $f(A) = \{f(x) : x \in A\}$ and similarly for B and $A \cap B$.)

   Solution: First, we show $f(A \cap B) \subset f(A) \cap f(B)$. Suppose that $y \in f(A \cap B)$. Then by definition, there is an $x \in A \cap B$ with $f(x) = y$. Since $x \in A$, we then have $y \in f(A)$. Since $x \in B$, we have $y \in f(B)$. Thus $y \in f(A) \cap f(B)$.

   We finish by showing that $f(A) \cap f(B) \subset f(A \cap B)$. Suppose that $y \in f(A) \cap f(B)$. Since $y \in f(A)$, there is an $x_1 \in A$ with $f(x_1) = y$. Since $y \in f(B)$ there is an $x_2 \in B$ with $f(x_2) = y$. Since f is injective and $f(x_1) = f(x_2)$, we have $x_1 = x_2$ and hence $x_1 \in A \cap B$. It follows that $y = f(x_1) \in f(A \cap B)$.

2. (10 points) Suppose that A, B are nonempty bounded subsets of $\mathbb{R}$ and let $A+B$ denote the set $\{x+y : x \in A \text{ and } y \in B\}$. Prove that $\inf(A+B) = \inf(A) + \inf(B)$.

   Solution: Let $z = \inf(A) + \inf(B)$.(here z makes sense and is a real number, since A, B are nonempty and bounded so $\inf(A), \inf(B) \neq \pm\infty$) We need to show that 1.) z is a lower bound for $A+B$, and that 2.) If w is a lower bound for $A+B$ then $w \leq z$.

   For 1.), suppose that $t \in A+B$. Then, by definition, there is an $x \in A$ and a $y \in B$ with $x+y=t$. Since $\inf(A)$ and $\inf(B)$ are lower bounds for A, B respectively, we have $\inf(A) \leq x$ and $\inf(B) \leq y$. It follows that $z = \inf(A) + \inf(B) \leq x+y = t$.

   Finishing with 2.) suppose that w is a lower bound for $A+B$. For contradiction, assume $w > z$. Since $\inf(A) + \frac{w-z}{2} > \inf(A)$ and $\inf(A)$ is the *greatest* lower bound for A, we have that $\inf(A) + \frac{w-z}{2}$ is not a lower bound for A. In other words, there is an $x \in A$ with $x < \inf(A) + \frac{w-z}{2}$. Similarly, there is a $y \in B$ with $y < \inf(B) + \frac{w-z}{2}$. Then

$$x + y < \inf(A) + \inf(B) + \frac{w-z}{2} + \frac{w-z}{2} = z + w - z = w.$$

   So, w is not a lower bound for $A+B$.

3. (10 points) Let $(a_n)_{n=1}^{\infty}$ be a real-valued sequence, and let $y, z \in \mathbb{R}$. Suppose that $\lim_{n\to\infty} a_n = y$ and $\lim_{n\to\infty} a_n = z$. Prove that $y = z$ (in other words the limit of a sequence is unique). (*hint:* prove this using the definition of a limit.)

   Solution: Suppose, for contradiction, that $y \neq z$. Then $y - z \neq 0$ so $|y - z|/2 > 0$. Since $\lim_{n\to\infty} a_n = y$, we may apply the definition of a limit with $\epsilon = \frac{|y-z|}{2}$, to see that there is an $N_1 \in \mathbb{N}$ such that $|a_n - y| < |y - z|/2$ for every $n \geq N_1$. Similarly, there is an $N_2 \in \mathbb{N}$ so that $|a_n - z| < |y - z|/2$ for every $n \geq N_2$. Then, using the triangle inequality, we conclude that

$$\begin{aligned} |y - z| = |y - a_{\max(N_1,N_2)} + a_{\max(N_1,N_2)} - z| &\leq |y - a_{\max(N_1,N_2)}| + |a_{\max(N_1,N_2)} - z| \\ &< \frac{|y-z|}{2} + \frac{|y-z|}{2} = |y - z|, \end{aligned}$$

   which is a contradiction.

4. (10 points) Let $(b_n)_{n=1}^{\infty}$ be a real-valued sequence, and $b \in \mathbb{R}$ with $\lim_{n\to\infty} b_n = b$. Suppose that $b > 0$ and that for every n, $b_n > 0$. Show that there is a real number $m > 0$ such that for every $n \in \mathbb{N}, b_n \geq m$. (*hint:* you can prove this using limit theorems, or by using the definition of a limit.)

   Solution 1:

   Since $\lim_{n\to\infty} b_n = b$, and $b \neq 0$, and for every n, $b_n \neq 0$, we may apply a limit theorem to see that $\lim_{n\to\infty} \frac{1}{b_n} = \frac{1}{b}$. Another limit theorem tells us that convergent sequences are bounded, and so there is a real number $M > 0$ such that $\frac{1}{b_n} \leq M$ for every $n \in \mathbb{N}$. Since $b_n > 0$ for every n, we conclude that $0 < \frac{1}{M} \leq b_n$ for every n.

   Solution 2:

   Since $b/2 > 0$ and $\lim_{n\to\infty} b_n = b$, there is an $N \in \mathbb{N}$ such that $|b_n - b| < b/2$ for every $n \geq N$. Then for every $n \geq N$, we have $b_n > \frac{b}{2}$ (since $b = |b| \leq |b - b_n| + |b_n| < b/2 + b_n$). Since, for every n we have $b_n \neq 0$, it follows that $m = \min(b_1, \ldots, b_N, \frac{b}{2}) > 0$ and $b_n \geq m$ for every n.

Extra Scratch Paper: