Math 131a **Final Exam**
Lecture 2
Spring 2009

Name: ____________________

Instructions:

- **There are 8 problems. Make sure you are not missing any pages.**
- **Unless stated otherwise (or unless it trivializes the problem), you may use without proof anything proven in the sections of the book covered by this test (excluding the exercises).**
- **Give complete, convincing, and clear answers (or points will be deducted).**
- **No calculators, books, or notes are allowed.**
- **Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Suppose that A is a nonempty bounded subset of $\mathbb{R}$ and that c is a real number with $c > 0$. Letting $B = \{c \cdot a : a \in A\}$ prove that $\sup B = c \cdot \sup A$.

Solution:

Let $y = c \sup A$. We need to show that i.) y is an upper bound for B and that ii.) if $x < y$ then x is not an upper bound for B.

For i.), suppose that $b \in B$. Then $b = ca$ for some $a \in A$. Since $\sup A$ is *an* upper bound for A and $c \geq 0$, we obtain $b = ca \leq c \sup A = y$, so y is an upper bound for B.

For ii.) suppose $x < y$. Again using $c > 0$, we have $x/c < y/c = \sup A$. Since $\sup A$ is the *least* upper bound, we have that x/c is not an upper bound for A and so there is an $a \in A$ with $a > x/c$. But, then $x < ca \in B$ and so x is not an upper bound for B.

2. (10 points) Suppose that $\sum_{n=1}^{\infty} a_n$ is convergent, that $a_n \geq 0$ for every n, and that $p > 1$ is a real number. Prove that $\sum_{n=1}^{\infty} a_n^p$ is convergent.

Solution:

Since $\sum_{n=1}^{\infty} a_n$ is convergent, we know (by a theorem in the book) that $\lim_{n\to\infty} a_n = 0$. Thus, there is an $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|a_n| < 1$. This gives $|a_n^p| = |a_n||a_n|^{p-1} < |a_n|1^{p-1} = a_n$ for all $n \geq N$. Thus, since $\sum_{n=1}^{\infty} a_n$ converges, the comparison test tells us that $\sum_{n=1}^{\infty} a_n^p$ converges.

3. (10 points) Suppose that $a, b \in \mathbb{R}$ and that $f : [a, b] \to \mathbb{R}$ is continuous. Let $C = \{f(x) : x \in [a, b]\}$ be the range of f, and suppose that $(y_n)_{n=1}^{\infty}$ is a convergent sequence with $y_n \in C$ for every n. Letting $y = \lim_{n\to\infty} y_n$ is it necessarily true that $y \in C$? Prove your answer.

Solution:

Yes, we do have $y \in C$. For each n, $y_n \in C$, so by definition of C there is some $x_n \in [a, b]$ with $f(x_n) = y_n$. The sequence $(x_n)_{n=1}^{\infty}$ is bounded, so by the Bolzano-Weierstrass theorem it has some convergent subsequence $(x_{n_k})_{k=1}^{\infty}$. Let $x = \lim_{k\to\infty} x_{n_k}$. Then, one can check that $x \in [a, b]$ (otherwise take $\epsilon = \max(x - b, a - x)$ to contradict the fact that each $x_{n_k} \in [a, b]$). Since f is continuous and $\lim_{k\to\infty} x_{n_k} = x$, a theorem from class (or the definition given in the book) tells us that $f(x) = \lim_{k\to\infty} f(x_{n_k}) = \lim_{k\to\infty} y_{n_k}$. But, since every subsequence of a convergent subsequence converges to the limit of the original sequence, this gives $f(x) = y$. So $y \in C$.

4. (10 points) Suppose that $g : \mathbb{R} \to \mathbb{R}$ is a bounded function, and consider the function $f : \mathbb{R} \to \mathbb{R}$ defined $f(x) = x^3 g(x)$. Is f necessarily differentiable at 0? Prove your answer.

Solution:

Yes, f is differentiable at 0 and $f'(0) = 0$. To see this let $B > 0$ satisfy $|g(x)| \leq B$ for every $x \in \mathbb{R}$ (guaranteed by boundedness). Then

$$\lim_{x \to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x \to 0} \frac{x^3 g(x)}{x} = \lim_{x \to 0} x^2 g(x).$$

Then for any $\epsilon > 0$, we can take $\delta = \sqrt{\frac{\epsilon}{B}}$ and

$$|x^2 g(x) - 0| < \frac{\epsilon}{B} |g(x)| \leq \frac{\epsilon}{B} B = \epsilon$$

whenever $0 < |x - 0| < \delta$.

5. (10 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $|f(x) - f(y)| \le |x - y|^2$ for every $x, y \in \mathbb{R}$. Prove that f is constant.

Solution:

To prove that f is constant, it will suffice (by a corollary to the mean value theorem in the book) to show that $f'(x) = 0$ for every $x \in \mathbb{R}$. But, letting $\epsilon > 0$, we can take $\delta = \epsilon$ and

$$\left| \frac{f(y) - f(x)}{y - x} - 0 \right| \le \frac{|y - x|^2}{|y - x|} = |y - x| < \epsilon$$

whenever $0 < |x - y| < \delta$. Thus, $f'(x) = \lim_{y \to x} \frac{f(y) - f(x)}{y - x} = 0$ for every x.

6. (10 points) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is differentiable, and that f' is bounded. Prove that f is uniformly continuous on $\mathbb{R}$.

Solution:

Let $\epsilon > 0$ we need to find $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $|x - y| < \delta$. Since f' is bounded, we may choose $B > 0$ such that $f'(c) \leq B$ for every c. Then, taking $\delta = \frac{\epsilon}{B}$ the mean value theorem tells us that for some c strictly between x and y, we have

$$|f(x) - f(y)| = |f'(c)(x - y)| \leq B|x - y| < B\frac{\epsilon}{B} = \epsilon$$

whenever $|x - y| < \delta$.

7. (10 points) Let $f : [0,2] \to \mathbb{R}$ be the function defined $f(1) = 1$ and $f(x) = 0$ for $x \neq 1$. Find (using only their definitions–not any theorems) the upper and lower Darboux integrals $U(f)$ and $L(f)$ on the interval $[0,2]$.

Solution:

First we look at the easier $L(f)$. For any partition $P = \{t_0, \ldots, t_n\}$ of $[0,2]$ we have

$$L(f,P) = \sum_{j=1}^{n} m(f,[t_{j-1},t_j])(t_j - t_{j-1}) = \sum_{j=1}^{n} 0(t_j - t_{j-1}) = 0$$

since every interval $[t_{j-1}, t_j]$ contains an x with $f(x) = 0$. Thus,

$$L(f) = \sup\{L(f,P) \text{ such that } P \text{ is a partition of } [0,2]\} = \sup\{0\} = 0.$$

For $U(f)$ let $P = \{t_0, \ldots, t_n\}$ be a partition of $[0,2]$. For each j we either have $M(f,[t_{j-1},t_j])) = 0$ or $= 1$ depending on whether $1 \in [t_{j-1}, t_j]$ or not. Then

$$U(f,P) = \sum_{j=1}^{n} M(f,[t_{j-1},t_j])(t_j - t_{j-1}) \geq \sum_{j=1}^{n} 0(t_j - t_{j-1}) = 0$$

So 0 is a lower bound for $\{U(f,P)$ such that P is a partition of $[0,2]\}$. If $x > 0$ then consider the partition $Q = \{0, 1 - \epsilon/2, 1 + \epsilon/2, 2\}$ where $\epsilon < \min(x, 2)$. Then

$$U(f,Q) = 0 \cdot (1 - \epsilon/2) + 1\epsilon + 0 \cdot (1 - \epsilon/2) = \epsilon < x.$$

Thus, 0 is the *greatest* lower bound for $\{U(f,P)$ such that P is a partition of $[0,2]\}$ and so $U(f) = 0$.

8. (10 points) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Consider the function $F : [2, \infty) \to \mathbb{R}$ defined

$$F(x) = \int_{x-1}^{x^2} f(t)\ dt.$$

Show that F is differentiable on $(2, \infty)$ and compute F'.

Solution:

By a theorem in the book, we have $F(x) = G(x^2) - G(x - 1)$ where $G(y) = \int_0^y f(t)\ dt$. Since f is continuous we have, by the Fundamental Theorem of Calculus 2, that G is differentiable on $(0, \infty)$ and $G'(y) = f(y)$ for every y. Thus, by the chain rule F is differentiable on $[2, \infty)$ with $F'(x) = 2xf(x^2) - f(x - 1)$.

Extra Scratch Paper:

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