SHORT NOTES ON COUNTABILITY

R. OBERLIN

We will say that a set B is *countable* if there exists a surjective function $f : \mathbb{N} \to B$ from the natural numbers onto B (this is equivalent to and sometimes slightly more convenient than the more conventional definition "B is countable if B has the same cardinality as $\mathbb{N}$ or if B is finite"; also some authors use the different definition "B is countable if it has the same cardinality as $\mathbb{N}$"). By convention, we will also say that the empty set is countable.

One obviously countable set is $\mathbb{N}$ itself, since the identity map $f(x) = x$ is a surjection from $\mathbb{N}$ onto $\mathbb{N}$. Next, we see that the integers are countable since the function defined $f(n) = (n-1)/2$ for odd numbers n and $f(n) = -n/2$ for even numbers n surjectively maps $\mathbb{N}$ onto $\mathbb{Z}$. This is an example of the slightly counterintuitive phenomenon that "an infinite set may have the same cardinality as a proper subset of itself."

More surprisingly, the set $\mathbb{N}\times\mathbb{N}$ is countable. To construct the surjective function $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ picture: $\mathbb{N} \times \mathbb{N}$

$$\begin{array}{lllll}
(1,1) & (1,2) & (1,3) & (1,4) & \dots \\
(2,1) & (2,2) & (2,3) & (2,4) & \dots \\
(3,1) & (3,2) & (3,3) & (3,4) & \dots \\
(4,1) & (4,2) & (4,3) & (4,4) & \dots \\
\dots & & & &
\end{array}$$

Then, each point $(x, y) \in \mathbb{N}\times\mathbb{N}$ lies on the $(x+y-1)'$th diagonal above. We define f as the function which counts the first diagonal, then counts the second diagonal (in the downward direction), then counts the third diagonal (in the downward direction), and so on and so forth; i.e. $f(1) = (1,1), f(2) = (1,2), f(3) = (2,1), f(4) = (1,3), f(5) = (2,2), f(6) = (3,1), \dots$. Since it takes $1 + 2 + \ldots + m = \frac{m(m+1)}{2}$ numbers to count the first m diagonals, and since any (x, y) is the x'th element on the $(x + y - 1)'$th diagonal, one can see that

$$(x, y) = f\left(x + \frac{(x+y-2)(x+y-1)}{2}\right)$$

and so f is surjective.

It then follows that if B_1, B_2 are any countable sets, then $B_1 \times B_2$ is countable. Indeed let $g_i : \mathbb{N} \to B_i$ be surjective functions for $i = 1, 2$. Let $\pi_i : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be defined $\pi_i(n_1, n_2) = n_i$ for $i = 1, 2$. Finally, set $h(n) = (g_1(\pi_1(f(n))), g_2(\pi_2(f(n))))$ where f is the function defined in the preceding paragraph. Then, one can check that $h : \mathbb{N} \to B_1 \times B_2$ is surjective. By induction, one can then check that for any countable set B and finite k, the set B^k is countable.

Suppose that Ω is a set whose elements are sets, that Ω is countable, and that A is countable for every $A \in \Omega$. Then it is a corollary of the previous paragraph that $\bigcup_{A\in\Omega} A$ is countable (i.e. any "countable union of countable sets" is countable). To see this, let $g : \mathbb{N} \to \Omega$ be surjective, and for each $n \in \mathbb{N}$ let $h_n : \mathbb{N} \to g(n)$

be surjective (these exist by the countability of each $g(n) \in \Omega$). Defining $i : \mathbb{N} \times \mathbb{N} \to \bigcup_{A\in\Omega} A$ by $i(n_1, n_2) = g_{n_1}(n_2)$, one can check that i is surjective and so $i \circ f : \mathbb{N} \to \bigcup_{A\in\Omega} A$ is surjective.

From the previous paragraph, it follows that the rational numbers are countable, and we will go a bit further than this. A real number x is *algebraic* if for some $n \in \mathbb{N}$ there exists an $(n+1)$-tuplet $(a_0, \ldots, a_n) \in \mathbb{Q}^{n+1}$ with at least one of the a_i's nonzero and with $a_0 + a_1x + \ldots + a_nx^n = 0$ (i.e. x is algebraic if it is the root of a nonzero polynomial with rational coefficients). The set of algebraic numbers is countable, as we now show. It is a fact from algebra that each nonzero polynomial of degree n has at most n roots, so in particular each set $A_{(a_0,\ldots,a_n)} = \{x \in \mathbb{R} : a_0 + a_1x + \ldots + a_nx^n = 0\}$ is finite and hence countable (provided at least one a_i is nonzero). Since $\mathbb{Q}$ is countable, we have $\mathbb{Q}^{n+1}$ countable, and thus any subset of $\mathbb{Q}^{n+1}$ is countable. It follows that $\Omega_n = \{A_{(a_0,\ldots,a_n)} : (a_0, \ldots, a_n) \in \mathbb{Q}^{n+1}$ and $a_i \neq 0$ for some i$\}$ is countable, and so $\bigcup_{A\in\Omega_n} A$ is countable. Then, since $\Omega = \{\bigcup_{A\in\Omega_n} A : n \in \mathbb{N}\}$ is countable we have $\bigcup_{B\in\Omega} B$ countable. But this last set is simply the set of algebraic numbers.

We will now see that the set of real numbers $\mathbb{R}$ is uncountable (the point of this discussion is that the vast majority of real numbers are not algebraic, i.e. they are *transcendental*). In "Short notes on basic math language" we saw that the set whose elements are the subsets of $\mathbb{N}$, which we will denote $\mathfrak{N}$ is uncountable. Consider a function $g : \mathbb{R} \to \mathfrak{N}$ defined as follows. If $x \in \mathbb{R}$ and any digit in the decimal expansion of x to the right of the decimal point is not in $\{1, 2\}$ then set $g(x) = \emptyset$, for every other $x \in \mathbb{R}$ (i.e. those numbers whose decimal expansion to the right of the decimal point is all 1's and 2's) set $g(x) = \{n \in \mathbb{N} :$ the n'th digit to the right of the decimal point in the decimal expansion of x is $2\}$. Then, g is defined unambiguously (since there are no issues with trailing 999999 vs 000000) and g maps $\mathbb{R}$ surjectively onto $\mathfrak{N}$. If $\mathbb{R}$ was countable with $f : \mathbb{N} \to \mathbb{R}$ surjective, then $g \circ f : \mathbb{N} \to \mathfrak{N}$ would be surjective, and so $\mathfrak{N}$ would be countable (a contradiction). Thus $\mathbb{R}$ is uncountable.