Math 131a
Lecture 2
Spring 2008

Midterm 2

Name: ______________________________

Instructions:

- There are 4 problems. Make sure you are not missing any pages.
- Give complete, convincing, and clear answers (or points will be deducted).
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Let A be a nonempty subset of $\mathbb{R}$ that is bounded above and let $\alpha = \sup A$. If $\alpha \notin A$, prove that α is a limit point of A.

Solution:

We suppose that α is not a limit point of A and show that $\alpha \neq \sup A$. Since α is not a limit point of A, there exists a $\delta > 0$ such that $(\alpha - \delta, \alpha + \delta) \cap A \subset \{\alpha\}$. Since $\alpha \notin A$, we thus have $(\alpha - \delta, \alpha + \delta) \cap A = \emptyset$. We may assume that α is an upper bound for α, since otherwise it certainly can't be the least upper bound and we're done. But, since α is an upper bound for A and $(\alpha - \delta, \alpha + \delta) \cap A = \emptyset$, we have that $\alpha - \delta$ is an upper bound for A, and so α is not the *least* upper bound for A.

2. (10 points) Let $\{a_k\}_{k=1}^{\infty}$ be a sequence of real numbers which converges to a point $a \in \mathbb{R}$. Consider the sequence $\{s_n\}_{n=1}^{\infty}$ defined

$$s_n = \frac{a_1 + \cdots + a_n}{n}.$$

Show that $\{s_n\}_{n=1}^{\infty}$ converges to a.

Solution:

Let $\epsilon > 0$. Since $\lim_{k\to\infty} a_k = a$, we may find an n_1 such that $|a_k - a| < \epsilon/2$ for $k \geq n_1$. Choose n_2 large enough so that $|a_1 - a + \ldots + a_{n_1} - a|/n_2 < \epsilon/2$. Then for $n > \max(n_1, n_2)$, we have

$$\begin{aligned}
|s_n - a| &= \left|\frac{a_1 + \cdots + a_n}{n} - a\right| \\
&= \left|\frac{a_1 - a + \cdots + a_n - a}{n}\right| \\
&\leq \frac{|a_1 - a + \ldots + a_{n_1} - a|}{n} + \frac{|a_{n_1+1} - a| + \ldots + |a_n - a|}{n} \\
&\leq \frac{|a_1 - a + \ldots + a_{n_1} - a|}{n_2} + \frac{n\frac{\epsilon}{2}}{n} \\
&< \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon
\end{aligned}$$

3. (10 points) Let $E \subset \mathbb{R}$ and let F be the set of limit points of E. Show that F is closed.

Solution:

We need to show that F^c is open. Suppose $x \in F^c$. Then x is not a limit point for E so there exists a $\delta > 0$ such that $N_\delta(x) \cap E \subset \{x\}$. We claim that $N_\delta(x) \subset F^c$ and hence F^c is open. Suppose $y \in N_\delta(x)$ and $y \neq x$. Let $\delta' = \min(|y-x|, \delta - |y-x|)$, and consider $N_{\delta'}(y)$. Since $\delta' \leq |y-x|$, we have $x \notin N_{\delta'}(y)$. Since $\delta' \leq \delta - |y-x|$, it follows from the triangle inequality that $N_{\delta'}(y) \subset N_\delta(x)$. Thus $N_{\delta'}(y) \cap E = \emptyset$, and so y is not a limit point for E, which means that $y \in F^c$.

4. (10 points) Show (you are only allowed to use the definitions and the triangle inequality) that every compact subset of $\mathbb{R}$ is closed.

Solution:

Suppose $K \subset \mathbb{R}$ is compact. We need to show that K^c is open. Suppose $x \in K^c$. Since $x \notin K$, we have $|y - x| > 0$ for each $y \in K$, and so $\{N_{\frac{|y-x|}{2}}(y)\}_{y \in K}$ is an open cover of K. Since K is compact, there is a finite subcover $\{N_{\frac{|y_1-x|}{2}}(y_1), \ldots, N_{\frac{|y_n-x|}{2}}(y_n)\}$. Set $\delta = \min(\frac{|y_1-x|}{2}, \ldots, \frac{|y_n-x|}{2})$ and consider $N_\delta(x)$. If $y \in K$ then $|y - y_i| < |y_i - x|/2$ for some i. Thus $|y_i - x| \leq |y_i - y| + |y - x| < |y_i - x|/2 + |y - x|$, and so $|y - x| > |y_i - x|/2 \geq \delta$. Hence $N_\delta(x) \subset K^c$.

Extra Scratch Paper: