Math 131a
Lecture 2
Spring 2008

Midterm 1

Name: ____________________

Instructions:

- There are 4 problems. Make sure you are not missing any pages.
- Give complete, convincing, and clear answers (or points will be deducted).
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (10 points) Let $f : A \to B$ and $g : B \to A$ be functions satisfying $(g \circ f)(x) = x$ for all $x \in A$. Prove that f is one-to-one. Must f be onto B? Justify your answer.

Solution:

First we prove that f is one-to-one. Suppose $x_1, x_2 \in A$ and $f(x_1) = f(x_2)$. We need to show that $x_1 = x_2$. Since $f(x_1) = f(x_2)$, we have $(g \circ f)(x_1) = g(f(x_1)) = g(f(x_2)) = (g \circ f)(x_2)$. But, by our assumption that $(g \circ f)(x) = x$, this implies that $x_1 = (g \circ f)(x_1) = (g \circ f)(x_2) = x_2$.

It is not necessarily the case that f is onto B. For example, we may take $A = \{1\}$, $B = \{1, 2\}$, $f = \{(1, 1)\}$ and $g = \{(1, 1), (2, 1)\}$. Then f does not map A onto B, but $g \circ f(x) = x$ for every $x \in A$.

2. (10 points) Let J be a nonempty subset of $\mathbb{R}$ that has the following properties:
   (i) J is bounded
   (ii) $\sup J \in J$
   (iii) $\inf J \notin J$
   (iv) If $x, y \in J$ with $x < y$ then $t \in J$ for every t satisfying $x < t < y$.

   Prove that J is the interval $(\inf J, \sup J]$.

Solution:

We need to show that a.) $J \subset (\inf J, \sup J]$, and *b*.) that $(\inf J, \sup J] \subset J$.

For a.), let $t \in J$. Since $\inf J$ is a lower bound for J and $\sup J$ is an upper bound for J, we have $\inf J \leq t \leq \sup J$. Since $\inf J \notin J$, we also have $\inf J < t \leq \sup J$ and so $t \in (\inf J, \sup J]$.

For b.), let $t \in (\inf J, \sup J]$. If $t = \sup J$, we have $t \in J$ by (ii), and so we may assume that $\inf J < t < \sup J$. Since $t > \inf J$ and $\inf J$ is the *greatest* lower bound, we see that t is not a lower bound for J and so there is an $x \in J$ with $x < t$. Since $t < \sup J$ and $\sup J$ is the *least* upper bound, we see that t is not an upper bound for J and so there is a $y \in J$ with $y > t$. Thus $x < t < y$, and so by property (iv), $t \in J$.

3. (10 points) Let A and B be nonempty subsets of the positive real numbers which are bounded above. Prove that $\sup\{a+b : a \in A, b \in B\} = (\sup A) + (\sup B)$.

Solution:

We need to show that (i) $\sup A + \sup B$ is an upper bound for $\{a+b : a \in A, b \in B\}$, and (ii) if $\alpha < \sup A + \sup B$ then α is not an upper bound for $\{a+b : a \in A, b \in B\}$.

For (i), let $a \in A$ and $b \in B$. Then $a \leq \sup(A)$ and $b \leq \sup B$, so $a+b \leq \sup A + b \leq \sup A + \sup B$.

For (ii), suppose $\alpha < \sup A + \sup B$ and let $\epsilon = \sup A + \sup B - \alpha > 0$. Since $\sup A$ is the *least* upper bound, we may find $a \in A$ with $a > \sup A - \epsilon/4$. Since $\sup B$ is the *least* upper bound, we may find $b \in B$ with $b > \sup B - \epsilon/4$. Then $a+b > \sup A + \sup B - \epsilon/2 > \sup A + \sup B - \epsilon = \alpha$. So α is not an upper bound for $\{a+b : a \in A, b \in B\}$.

4. (10 points) For each positive integer k let

$$\mathbb{N}^k = \{(x_1, \ldots, x_k) : x_1, \ldots, x_k \in \mathbb{N}\}$$

be the set of ordered k-tuplets of positive integers. Prove that for every positive integer k, $\mathbb{N}^k$ is countable. You are allowed to use, without proof, the fact that $\mathbb{N} \times \mathbb{N}$ is countable. You are also allowed to use, without proof, the fact that if $f : A \to B$ and $g : B \to C$ are both bijective, then $g \circ f : A \to C$ is bijective. *Hint:* Try using induction on k.

Solution:

By induction it suffices to show that (i) $\mathbb{N}^1$ is countable and that (ii) if $\mathbb{N}^k$ is countable then $\mathbb{N}^{k+1}$ is countable.

For (i), it is clear that $f : \mathbb{N} \to \mathbb{N}^1$ defined by $f(n) = (n)$ is a bijection, and so $\mathbb{N}^1$ is countable.

For (ii) assume that $\mathbb{N}^k$ is countable. Then, there exists a bijective $g : \mathbb{N} \to \mathbb{N}^k$.

Consider the function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}^{k+1}$ defined by $f(n, m) = (n, y_2, \ldots, y_{k+1})$ where $(y_2, \ldots, y_{k+1}) = g(m)$. We claim that f is bijective. To see that f is onto, let $(y_1, \ldots, y_{k+1}) \in \mathbb{N}^{k+1}$. Since g is onto, there exists $m \in N$ with $g(m) = (y_2, \ldots, y_{k+1})$, and so $f(y_1, m) = (y_1, \ldots, y_{k+1})$. To see that f is one-to-one, suppose $f(n_1, m_1) = (y_1, \ldots, y_{k+1}) = f(n_2, m_2)$. Then $g(m_1) = (y_2, \ldots, y_{k+1}) = g(m_2)$. Since g is one-to-one, this implies that $m_1 = m_2$. But we also have $n_1 = y_1 = n_2$ and so $(n_1, m_1) = (n_2, m_2)$.

Since $\mathbb{N} \times \mathbb{N}$ is countable, there exists a bijective $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$. Since f is bijective and h is bijective, $f \circ h : \mathbb{N} \to \mathbb{N}^{k+1}$ is bijective, and so $\mathbb{N}^{k+1}$ is countable.

Extra Scratch Paper: