Math 131a Lecture 2
Homework 2 – Solution

Section 1.4 #18

We have X, Y nonempty and $f : X \times Y \to \mathbb{R}$ nonnegative. Let $\beta = \sup\{f(x,y) : (x,y) \in X \times Y\}$ and for each $x \in X$ let $F(x) = \sup\{f(x,y) : y \in Y\}$. We will show that $\beta = \sup\{F(x) : x \in X\}$. To do this it suffices to show that (i) β is an upper bound for $\{F(x) : x \in X\}$, and (ii) if $\alpha < \beta$ then α is not an upper bound for $\{F(x) : x \in X\}$.

We start with (i). Let $x \in X$. We need to show that $F(x) \leq \beta$. But $F(x)$ is the *least* upper bound for $\{f(x,y) : y \in Y\}$, so if β is an upper bound for $\{f(x,y) : y \in Y\}$ then $F(x) \leq \beta$ and we're done. For every $y \in Y$ we have $(x,y) \in X \times Y$ and so, since β is an upper bound for $\{f(x,y) : (x,y) \in X \times Y\}$, we have $f(x,y) \leq \beta$. Thus β is an upper bound for $\{f(x,y) : y \in Y\}$.

To see (ii), let $\alpha < \beta$. Since β is the *least* upper bound, α is not an upper bound for $\{f(x,y) : (x,y) \in X \times Y\}$ and so there exists (x_0, y_0) such that $f(x_0, y_0) > \alpha$. We claim that $F(x_0) > \alpha$ and so α is not an upper bound for $\{F(x) : x \in X\}$. Indeed, by definition, $F(x_0)$ is an upper bound for $\{f(x_0, y) : y \in Y\}$ and so $F(x_0) \geq f(x_0, y_0) > \alpha$.

The fact that $\beta = \sup\{G(y) : y \in Y\}$ follows by an analogous argument.