Math 131a **Final Exam**
Lecture 2
Spring 2008

Name: ____

Instructions:

- **There are 7 problems. Make sure you are not missing any pages.**
- **Give complete, convincing, and clear answers (or points will be deducted).**
- **No calculators, books, or notes are allowed.**
- **Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
Total:	70	

1. (10 points) Let X, Y be nonempty sets and let f be a bounded, nonnegative, real-valued function defined on $X \times Y$. For each $x \in X$ define

$$G(x) = \sup\{f(x,y) : y \in Y\}.$$

Prove that $\sup\{G(x) : x \in X\} = \sup\{f(x,y) : (x,y) \in X \times Y\}$.

Solution:

First, we prove that $\sup\{G(x) : x \in X\} \leq \sup\{f(x,y) : (x,y) \in X \times Y\}$. By definition of *least* upper bound, it suffices to show that $\sup\{f(x,y) : (x,y) \in X \times Y\}$ is an upper bound for $\{G(x) : x \in X\}$. Suppose $x_0 \in X$. Since $(x_0, y) \in X \times Y$ for every $y \in Y$, we have that $\sup\{f(x,y) : (x,y) \in X \times Y\}$ is an upper bound for $\{f(x_0,y) : y \in Y\}$ and so $G(x_0) = \sup\{f(x_0,y) : y \in Y\} \leq \sup\{f(x,y) : (x,y) \in X \times Y\}$.

Now, we prove that $\sup\{G(x) : x \in X\} \geq \sup\{f(x,y) : (x,y) \in X \times Y\}$. Again, it suffice to show that $\sup\{G(x) : x \in X\}$ is an upper bound for $\{f(x,y) : (x,y) \in X \times Y\}$. Suppose $(x_0, y_0) \in X \times Y$. Then $f(x_0, y_0) \leq \sup\{f(x_0,y) : y \in Y\} = G(x_0) \leq \sup\{G(x) : x \in X\}$.

2. (10 points) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}$ satisfying $|a_n - a_{n+1}| \geq c$ for some $c > 0$ and all $n \in \mathbb{N}$. Prove (using only the definition) that $\{a_n\}_{n=1}^{\infty}$ is not convergent.

Solution:

Suppose, for contradiction, that $\{a_n\}_{n=1}^{\infty}$ converges to some $p \in \mathbb{R}$. Since $c/2 > 0$, there exists an $n_0 \in \mathbb{N}$ such that $|a_n - p| < c/2$ for every $n \geq n_0$. For every $n \geq n_0$ we then have $|a_n - a_{n+1}| = |a_n - p + p - a_{n+1}| \leq |a_n - p| + |a_{n+1} - p| < c/2 + c/2 = c$. This contradicts the hypothesis that $|a_n - a_{n+1}| \geq c$.

3. (10 points) Let K be a compact subset of $\mathbb{R}$. Prove (using only the definitions) that K is bounded.

Solution:

We consider the open cover $\{N_1(x)\}_{x\in K}$ of K. Since K is compact, there is a finite subcover $\{N_1(x_1), \ldots, N_1(x_n)\}$. We set $M = 1 + \max(|x_1|, \ldots, |x_k|)$ and claim that K is bounded by M. Suppose $x \in K$. Then, since $\{N_1(x_1), \ldots, N_1(x_n)\}$ covers K, we have $x \in N_1(x_i)$ for some i. It follows that $|x| = |x - x_i + x_i| \leq |x - x_i| + |x_i| \leq 1 + |x_i| \leq M$.

4. (10 points) Suppose E is a subset of $\mathbb{R}$ and $f, g : E \to \mathbb{R}$ are continuous at $p \in E$. Define $h : E \to \mathbb{R}$ by $h(x) = \max\{f(x), g(x)\}$. Prove that h is continuous at p.

Solution:

Let $\epsilon > 0$. We need to find $\delta > 0$ such that $|h(x) - h(p)| < \epsilon$ for every $x \in E$ satisfing $|x - p| < \delta$. There are three possible cases: $f(p) > g(p), f(p) < g(p), f(p) = g(p)$.

We will first consider the case when $f(p) > g(p)$. Let $\epsilon_1 = \min(\epsilon, (f(p) - g(p))/2)$. Since f is continuous at p, there exists a $\delta_1 > 0$ such that $|f(x) - f(p)| < \epsilon_1$ for every $x \in E$ satisfying $|x - p| < \delta_1$. Since g is continuous at p, there exists a $\delta_2 > 0$ such that $|g(x) - g(p)| < \epsilon_1$ for every $x \in E$ satisfying $|x - p| < \delta_2$. Let $\delta = \min(\delta_1, \delta_2)$ and suppose $x \in E$ with $|x - p| < \delta$. Since $|x - p| < \delta \leq \delta_1$, we have $f(x) > f(p) - \epsilon > f(p) - (f(p) - g(p))/2) = (f(p) + g(p))/2$. Since $|x - p| < \delta \leq \delta_2$, we have $g(x) < g(p) + \epsilon < g(p) + (f(p) - g(p))/2) = (f(p) + g(p))/2$. Thus, $f(x) > g(x)$ and so $|h(x) - h(p)| = |\min(f(x), g(x)) - \min(f(p), g(p))| = |g(x) - g(p)| < \epsilon_1 < \epsilon$. For the second to last inequality, we again use the fact that $|x - p| < \delta_2$.

The case when $f(p) < g(p)$ follow by an argument analogous to that for the case $f(p) > g(p)$.

Finally suppose $f(p) = g(p)$. Find $\delta > 0$, as above, so that if $x \in E$ and $|x - p| < \delta$ we have $|f(x) - f(p)| < \epsilon$ and $|g(x) - g(p)| < \epsilon$. Let $x \in E$ with $|x - p| < \delta$. There are two possible cases: $f(x) \geq g(x)$ and $f(x) < g(x)$. In the former case, we have $|h(x) - h(p)| = |\min(f(x), g(x)) - \min(f(p), g(p))| = |g(x) - g(p)| < \epsilon$. In the latter case, we have $|h(x) - h(p)| = |\min(f(x), g(x)) - \min(f(p), g(p))| = |f(x) - f(p)| < \epsilon$.

5. (10 points) Suppose that E is a bounded subset of $\mathbb{R}$ and that $f : E \to \mathbb{R}$ is uniformly continuous on E. Prove that f is bounded on E. You are allowed to use theorems from the book (such as the Heine-Borel theorem).

Solution:

Since f is uniformly continuous on E, we may find a $\delta > 0$ such that $|f(x) - f(x')| < 1$ whenever $x, x' \in E$ and $|x - x'| < \delta$.

Clearly $\{N_\delta(x)\}_{x \in E}$ is an open cover of E. We claim that it also covers the closure of E, $\overline{E}$. Recall that $\overline{E}$ is the union of E with its limit points. Suppose y is a limit point of E. Then there is an $x \in E$ with $x \in N_\delta(y)$, so $y \in N_\delta(x)$.

We know that E is bounded, say by M. By an argument similar to that in the preceding paragraph, we have that $\overline{E}$ is bounded by $M + \delta$.

Since $\overline{E}$ is a closed bounded set of real numbers, it is compact (by the Heine-Borel theorem). Thus, $\overline{E}$ and hence E is covered by a finite subcollection $\{N_\delta(x_1), \ldots, N_\delta(x_n)\}$ of $\{N_\delta(x)\}_{x \in E}$.

Let $B = 1 + \max(|f(x_1)|, \ldots, |f(x_n)|)$. We claim that f is bounded by B. Indeed, suppose $x \in E$. Then $x \in N_\delta(x_i)$ for some i. Since $|x - x_i| < \delta$, we have $|f(x)| = |f(x) - f(x_i) + f(x_i)| \le |f(x) - f(x_i)| + |f(x_i)| < 1 + |f(x_i)| \le B$.

6. (10 points) Suppose $f : (-1,1) \to \mathbb{R}$ is bounded. Define $g : (-1,1) \to \mathbb{R}$ by $g(x) = x^3 f(x)$. Prove that g is differentiable at 0. What is $g'(0)$?

Solution:

We claim that g is differentiable at 0 and that $g'(0) = 0$. In otherwords, that $\lim_{x\to 0} \frac{g(x)-g(0)}{x-0} = 0$.

Let $\epsilon > 0$. We need to find $\delta > 0$ so that $|\frac{g(x)-g(0)}{x-0} - 0| < \epsilon$ whenever $0 < |x - 0| < \delta$. Since f is bounded on $(-1,1)$, we may find $M > 0$ such that $|f(x)| < M$ for every $x \in (-1,1)$. Let $\delta = \min(1, \epsilon/M)$ and suppose that $0 < |x - 0| < \delta$. Then $|\frac{g(x)-g(0)}{x-0} - 0| = |\frac{x^3 f(x)}{x}| = |x||x||f(x)| < 1\frac{\epsilon}{M}M = \epsilon$.

7. (10 points) Let $f : [a, b] \to \mathbb{R}$ be differentiable on (a, b). Suppose $c \in (a, b)$ is such that $f'(c) = 0$, f' is differentiable at c, and $f''(c) > 0$. Prove that f has a local minimum at c.

Solution:

Since $f''(c)/2 > 0$, we may find a $\delta_1 > 0$ so that $|\frac{f'(x)-f'(c)}{x-c} - f''(c)| < f''(c)$ whenever $x \in [a, b]$ and $0 < |x - c| < \delta_1$. Letting $\delta = \min(\delta_1, |c - a|, |c - b|)$, we then have $\frac{f'(x)}{x-c} = \frac{f'(x)-f'(c)}{x-c} > f''(c) - f''(c) = 0$ whenever $0 < |x - c| < \delta$. Thus, $f' \geq 0$ on $[c, c+\delta)$ and $f' \leq 0$ on $(c - \delta, c]$. It follows from the mean value theorem (see the Corollary in the book) that f is monotonically increasing on $[c, c + \delta)$ and monotonically decreasing on $(c - \delta, c]$. Thus $f(x) \geq f(c)$ for every $x \in N_\delta(x)$.

Extra Scratch Paper:

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