

Name: \_\_\_\_\_

**Instructions:**

- There are 4 problems. Make sure you are not missing any pages.
- Unless stated otherwise, you may use without proof anything proven in the sections of the book covered by this test (excluding the exercises).
- Give complete, convincing, and clear answers (or points will be deducted).
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
<b>Total:</b>	<b>40</b>	

1. (10 points) Let  $V$  be a vector space over  $\mathbb{R}$  with ordered basis  $\beta = \{v_1, v_2, v_3\}$ ; let  $\gamma$  be the ordered basis of  $\mathbb{R}^3$  defined  $\gamma = \{(1, 1, 0), (1, 0, 0), (0, 0, 1)\}$ ; let  $T : V \rightarrow \mathbb{R}^3$  be the linear transformation whose matrix representation is given

$$[T]_{\beta}^{\gamma} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Compute  $T(v_1 + 2v_2 + 3v_3)$ .

**Solution:**

By definition of matrix representation, we have  $[T(v_1)]_{\gamma} = (0, 0, 1)^T$  and so  $T(v_1) = 0(1, 1, 0) + 0(1, 0, 0) + 1(0, 0, 1) = (0, 0, 1)$ . Similarly  $T(v_2) = 0(1, 1, 0) + 1(1, 0, 0) + 0(0, 0, 1) = (1, 0, 0)$  and  $T(v_3) = 1(1, 1, 0) + 0(1, 0, 0) + 0(0, 0, 1) = (1, 1, 0)$ . Since  $T$  is linear, we thus have  $T(v_1 + 2v_2 + 3v_3) = T(v_1) + 2T(v_2) + 3T(v_3) = (0, 0, 1) + 2(1, 0, 0) + 3(1, 1, 0) = (5, 3, 1)$ .

2. (10 points) Suppose that  $V$  is a finite dimensional vector space, that  $T : V \rightarrow V$  and  $U : V \rightarrow V$  are linear transformations, and that their composition  $TU$  is invertible. Prove that  $T$  is invertible and that  $U$  is invertible.

**Solution:**

We know that any function is invertible if and only if it is one-to-one and onto. However, since  $V$  is finite dimensional, we know from Theorem 2.5 in the book that any linear transformation from  $V$  to  $V$  is one-to-one if and only if it is onto, and so we conclude that any linear transformation from  $V$  to  $V$  is invertible if it is one-to-one or onto. It thus suffices to show that  $T$  is onto and  $U$  is one-to-one.

Since  $TU$  is invertible, it is onto  $V$ , and so the range of  $TU$  is  $V$ . Clearly, the range of  $TU$  is contained in the range of  $T$  and so the range of  $T$  is  $V$  and hence  $T$  is onto.

Since  $TU$  is invertible, it is one-to-one. This implies that  $U$  is one-to-one. Indeed, suppose that  $U(x) = U(y)$ . Then  $TU(x) = TU(y)$  and since  $TU$  is one-to-one we conclude that  $x = y$ .

3. (10 points) Recall that the trace of an  $n \times n$  matrix  $A$  is defined  $\text{trace}(A) = \sum_{i=1}^n A_{i,i}$ . Suppose that  $B$  is an  $n \times n$  matrix and that  $Q$  is an invertible  $n \times n$  matrix. Prove that

$$\text{trace}(Q^{-1}BQ) = \text{trace}(B).$$

**Solution:**

We first claim that it suffices to show that  $\text{trace}(CD) = \text{trace}(DC)$  for each pair of  $n \times n$  matrices  $C, D$ . Indeed, once this is known we have  $\text{trace}(Q^{-1}BQ) = \text{trace}(BQQ^{-1}) = \text{trace}(BI_n) = \text{trace}(B)$ .

To finish, observe that  $\text{trace}(CD) = \sum_{i=1}^n (CD)_{ii} = \sum_{i=1}^n \sum_{j=1}^n C_{i,j}D_{j,i} = \sum_{j=1}^n \sum_{i=1}^n D_{j,i}C_{i,j} = \sum_{j=1}^n (DC)_{j,j} = \text{trace}(DC)$ .

4. (10 points) Suppose that  $V$  is a finite dimensional vector space and that  $T : V \rightarrow V$  is an invertible, diagonalizable, linear transformation. Prove that  $T^{-1}$  is diagonalizable.

**Solution:**

Since  $T$  is diagonalizable, we know from Theorem 5.1 in the book that there is an ordered basis  $\beta = \{v_1, \dots, v_n\}$  for  $V$  such that  $v_i$  is an eigenvector for  $T$  for each  $i$ . Let  $\lambda_i$  be the eigenvalue associated with  $v_i$  for each  $i$ . Since  $T$  is invertible, we know that  $T$  is one-to-one and so the nullspace of  $T$  is  $\{0_V\}$  (by Theorem 2.5 in the book). Thus  $\lambda_i \neq 0$  for each  $i$  (or we would have  $0_V \neq v_i \in N(T)$ ). We claim that, for each  $i$ ,  $v_i$  is an eigenvector for  $T^{-1}$  with eigenvalue  $\lambda_i^{-1}$  and so (again by Theorem 5.1)  $T^{-1}$  is diagonalizable. Indeed  $v_i \neq 0_V$  and  $T(\lambda_i^{-1}v_i) = \lambda_i^{-1}T(v_i) = \lambda_i^{-1}\lambda_i v_i = v_i$ , and so  $T^{-1}(v_i) = \lambda_i^{-1}v_i$ .

Extra Scratch Paper: