Math 115a **Midterm 1**
Lecture 3
Winter 2009

Name: ______________________________

<u>Instructions:</u>

- **There are 4 problems. Make sure you are not missing any pages.**
- **Unless stated otherwise, you may use without proof anything proven in the sections of the book covered by this test (excluding the exercises).**
- **Give complete, convincing, and clear answers (or points will be deducted).**
- **No calculators, books, or notes are allowed.**
- **Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.**

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
Total:	40	

1. (a) (5 points) Let V be the set $\mathbb{R}^2$ with the usual vector addition operation $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and with the scalar multiplication operation $c(x_1, x_2) = (cx_1, x_2)$. Is V a vector space over $\mathbb{R}$? Justify your answer using only the definition of a vector space (i.e. do not use anything that is proven in the book).

Solution:

By definition, a vector space must satisfy $(c_1 + c_2)v = c_1v + c_2v$ for every $c_1, c_2 \in F$ and $v \in V$. However with the operations above $(0 + 0)(1, 1) = 0(1, 1) = (0, 1) \neq (0, 2) = 0(1, 1) + 0(1, 1)$ so V is not a vector space.

(b) (5 points) Let V be the set $\mathbb{R}^2$ with the usual vector addition operation $(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$ and with the scalar multiplication operation $c(x_1, x_2) = (cx_1, 3cx_2)$. Is V a vector space over $\mathbb{R}$? Justify your answer using only the definition of a vector space (i.e. do not use anything that is proven in the book).

Solution:

By definition, a vector space must satisfy $1v = v$ for every $v \in V$. However with the operations above $1(1, 1) = (1, 3) \neq (1, 1)$ so V is not a vector space.

2. (10 points) Let V be any vector space over a field F, and let W_1, W_2, and $W_1 \cup W_2$ be subspaces of V satisfying $W_1 \not\subset W_2$. Prove that $W_2 \subset W_1$.

Solution:

Suppose $x_2 \in W_2$; we need to show that $x_2 \in W_1$. Since $W_1 \not\subset W_2$ there is an $x_1 \in W_1$ so that $x_1 \notin W_2$. Since x_1, x_2 are in the vector space $W_1 \cup W_2$ we must have $x_1 + x_2 \in W_1 \cup W_2$. Notice that we can't have $x_1 + x_2 \in W_2$, indeed if this were the case then we could use the fact that W_2 is a vector space containing $-x_2$ to conclude that $x_1 = (x_1 + x_2) - x_2$ is in W_2 which would contradict the choice of x_1. Thus $x_1 + x_2$ is in $W_1 \cup W_2$ but not in W_2, and so we conclude that it is in W_1. But then, since W_1 is a vector space containing $-x_1$, we have $x_2 = (x_1 + x_2) - x_1 \in W_1$ which is what we were trying to prove.

3. (10 points) Let V, W be any vector spaces over a field F, let $\beta = \{v_1, \ldots, v_n\}$ be a basis for V, and let $T : V \to W$ be a one-to-one and onto linear transformation. Prove that $T(\beta)$ (which is the set $\{T(v_1), \ldots, T(v_n)\}$) is a basis for W.

Solution:

We need to show that a.)$T(\beta)$ is linearly independent and that b.)$W \subset \text{span}(T(\beta))$.

For b.) suppose that $w \in W$. Since T is onto, there is a $v \in V$ such that $T(v) = w$. Since β is a basis it spans V and so there are $a_1, \ldots, a_n \in F$ such that $v = a_1 v_1 + \ldots + a_n v_n$. Since T is linear, we then have $w = T(v) = T(a_1 v_1 + \ldots + a_n v_n) = a_1 T(v_1) + \ldots + a_n T(v_n)$ as required.

For a.) suppose $a_1 T(v_1) + \ldots + a_n T(v_n) = 0_W$; then we need to show that $a_1 = \ldots = a_n = 0$. Since T is linear, we have $T(a_1 v_1 + \ldots + a_n v_n) = 0_W$. Since $T(0_V) = 0_W$ and T is one-to-one, we have $a_1 v_1 + \ldots + a_n v_n = 0_V$. Since β is a basis it is linearly independent, implying that $a_1 = \ldots = a_n = 0$ as required.

4. (10 points) Let V be any vector space over a field F, and let W, Z be subspaces of V satisfying $W \cap Z = \{0_V\}$. Suppose that $\beta = \{w_1, w_2\}$ is a basis for W and $\gamma = \{z_1, z_2, z_3\}$ is a basis for Z. Prove that $\dim(\operatorname{span}(W \cup Z)) = 5$.

Solution:

By definition of dimension and the fact that $\beta \cap \gamma = \emptyset$ it suffices to show that $\delta = \{w_1, w_2, z_1, z_2, z_3\}$ is a basis for $\operatorname{span}(W \cup Z)$, in other words that a.)δ is linearly independent and that b.)$\operatorname{span}(W \cup Z) \subset \operatorname{span}(\delta)$.

For a.) suppose that $a_1w_1 + a_2w_2 + b_1z_1 + b_2z_2 + b_3z_3 = 0$. Then

$$a_1w_1 + a_2w_2 = -(b_1z_1 + b_2z_2 + b_3z_3).$$

The left hand side of the equation above is in W and the right hand side is in Z, and so both sides must be in $W \cap Z = \{0_V.\}$ Since β is linearly independent and $a_1w_1 + a_2w_2 = 0_V$, we have $a_1 = a_2 = 0$. Similarly $b_1 = b_2 = b_3 = 0$ as required.

Since $\operatorname{span}(\delta)$ is a subspace of V, Theorem 1.5 in the book implies that b.) will follow after observing that $W \cup Z \subset \operatorname{span}(\delta)$. However since $\beta \subset \delta$ then $W = \operatorname{span}(\beta) \subset \operatorname{span}(\delta)$. Similarly $Z \subset \operatorname{span}(\delta)$ as required.

Extra Scratch Paper: