Section 6.4 #11b

We know $\langle T(x), x\rangle = 0$ for every x and need to show that $T(x) = 0$ for every x. Following the hint, we see that

$$\begin{aligned}0 &= \langle T(x+y), x+y\rangle \\ &= \langle T(x), x\rangle + \langle T(y), y\rangle + \langle T(y), x\rangle + \langle T(x), y\rangle \\ &= 0 + 0 + \langle T(y), x\rangle + \langle T(x), y\rangle\end{aligned}$$

which tells us that $\langle T(x), y\rangle = -\langle T(y), x\rangle$ for every x, y. On the other hand, we have

$$\begin{aligned}0 &= \langle T(x+iy), x+iy\rangle \\ &= \langle T(x), x\rangle + \langle T(y), y\rangle + i\langle T(y), x\rangle - i\langle T(x), y\rangle \\ &= 0 + 0 + i\langle T(y), x\rangle - i\langle T(x), y\rangle\end{aligned}$$

which tells us that $\langle T(x), y\rangle = \langle T(y), x\rangle$ for every x, y. Adding these together, we conclude that $\langle T(x), y\rangle = 0$ for every x, y and in particular $\|T(x)\| = \sqrt{|\langle T(x), T(x)\rangle|} = 0$ implying that $T(x)$ for every x.