Section 1.5 #18 Suppose $u_1, \ldots, u_n \in S$ are distinct vectors; $a_1, \ldots, a_n \in F$; and $a_1u_1 + \ldots + a_nu_n = 0$. We need to show that $a_i = 0$ for $i = 1, \ldots, n$. Since no two vectors in S have the same degree, we may re-order the $u_1, \ldots, u_n$ so that $\deg(u_i) < \deg(u_j)$ whenever $i < j$. Assume that the conclusion is false, in other words that $a_i \neq 0$ for some i, and let i_0 be largest element of $\{i : a_i \neq 0\}$. Then

$$u_{i_0} = \sum_{j<i_0} \frac{-a_j}{a_{i_0}} u_j.$$

The right side of the equation above is a polynomial with degree strictly less than $\deg(u_{i_0})$, and so we conclude that $\deg(u_{i_0}) < \deg(u_{i_0})$, a contradiction.

Section 1.6 #8

First, one may observe via Gaussian elimination that if $a_1u_5 + a_2u_6 + a_3u_7 + a_4u_8 = 0$ then $a_1 = a_2 = a_3 = a_4 = 0$ and so $L = \{u_5, u_6, u_7, u_8\}$ is linearly independent. Next, since W is a subspace of $\mathbb{R}^5$ and $W \neq \mathbb{R}^5$ (because for example $(1, 1, 1, 1, 1) \notin W$), we have from Theorem 1.11 that W is finite-dimensional and $\dim(W) < \dim(\mathbb{R}^5) = 5$. By definition of finite-dimensional, W has a basis G. Applying the replacement theorem with L and G, we see that $4 \leq \dim(W)$ and so $\dim(W) = 4$. By Corollary 2b of the replacement theorem, we finally conclude that L is a basis for W.

An alternative way to see that $\dim(W) = 4$ is to note that if $w = (a_1, a_2, a_3, a_4, a_5) \in W$ then $a_1 = -a_2 - a_3 - a_4 - a_5$ and so $w = a_2(-1, 1, 0, 0, 0) + a_3(-1, 0, 1, 0, 0) + a_4(-1, 0, 0, 1, 0) + a_5(-1, 0, 0, 0, 1)$. Thus

$$W \subset \operatorname{span}(\{(-1, 1, 0, 0, 0), (-1, 0, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 0, 0, 0, 1)\}).$$

Since $\{(-1, 1, 0, 0, 0), (-1, 0, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 0, 0, 0, 1)\}$ is a linearly independent subset of W which generates W, it is a basis for W and so $\dim(W) = 4$.