Math 115a
Lecture 3
Winter 2009

Final Exam

Name: ______________________________

Instructions:

- There are 8 problems. Make sure you are not missing any pages.
- Unless stated otherwise, you may use without proof anything proven in the sections of the book covered by this test (excluding the exercises).
- Give complete, convincing, and clear answers (or points will be deducted).
- No calculators, books, or notes are allowed.
- Answer the questions in the spaces provided on the question sheets. If you run out of room for an answer, continue on the back of the page.

Question	Points	Score
1	10	
2	10	
3	10	
4	10	
5	10	
6	10	
7	10	
8	10	
Total:	80	

1. (10 points) Let V be a vector space, let $T_0 : V \to V$ be the zero transformation (that is, $T_0(v) = 0_v$ for every $v \in V$), and let $T : V \to V$ be any linear transformation. Prove that

$$T^2 = T_0 \Leftrightarrow \text{range}(T) \subset \text{nullspace}(T)$$

Solution:

$\Rightarrow$

Suppose $T^2 = T_0$ and $y \in \text{range}(T)$; we need to show that $y \in \text{nullspace}(T)$. By definition of range, there is an $x \in V$ such that $T(x) = y$. Since $T^2 = T_0$ we have $0_V = T^2(x) = T(T(x)) = T(y)$, and so $y \in \text{nullspace}(T)$.

$\Leftarrow$

Suppose that $\text{range}(T) \subset \text{nullspace}(T)$ and that $x \in V$. We need to show that $T^2(x) = 0_V$. Since $T(x) \in \text{range}(T)$, we have $T(x) \in \text{nullspace}(T)$ and so $T^2(x) = T(T(x)) = 0_V$.

2. (10 points) Let V be a vector space, and let $T : V \to V$ and $U : V \to V$ be linear transformations. Suppose that $TU = UT$, that λ is an eigenvalue for T, and that E_λ is the eigenspace for T associated with λ. Show that $U(E_\lambda) \subset E_\lambda$.

Solution:

Suppose that $y \in U(E_\lambda)$; we need to show that $y \in E_\lambda$. By definition of $U(E_\lambda)$ there is an $x \in V$ such that $T(x) = \lambda x$ and $U(x) = y$. Then $UT(x) = U(\lambda x) = \lambda U(x) = \lambda y$. Since $TU = UT$, we then have $T(y) = TU(x) = UT(x) = \lambda y$ and so $y \in E_\lambda$.

3. (10 points) Let V be a finite dimensional vector space, and let $T : V \to V$ and $U : V \to V$ be linear transformations. Suppose that $\{v_1, \dots, v_n\}$ is a basis for V such that v_i is an eigenvector for T and v_i is an eigenvector for U for every $i = 1, \dots, n$. Prove that $TU = UT$.

Solution:

Suppose that $w \in V$, we need to show that $TU(w) = UT(w)$. Since $\{v_1, \dots, v_n\}$ is a basis, we have $w = a_1v_1 + \ldots + a_nv_n$ for some $a_1, \dots, a_n \in F$. Let $\lambda_1, \dots, \lambda_n$ be the eigenvalues for T associated with $v_1, \dots, v_n$ respectively and let $\mu_1, \dots, \mu_n$ be the eigenvalues for U associated with $v_1, \dots, v_n$ respectively. Then for $i = 1, \dots, n$ we have $UT(v_i) = U(\lambda_i v_i) = \lambda_i U(v_i) = \lambda_i \mu_i v_i = \mu_i T(v_i) = T(\mu_i v_i) = TU(v_i)$. Thus

$$TU(w) = TU(\sum_{i=1}^{n} a_i v_i) = \sum_{i=1}^{n} a_i TU(v_i) = \sum_{i=1}^{n} a_i UT(v_i) = UT(\sum_{i=1}^{n} a_i v_i) = UT(w).$$

4. (10 points) Suppose that V is a finite dimensional inner product space with orthonormal basis $\{v_1, \ldots, v_n\}$. Using only the definitions of orthonormal bases and inner products, show that for every $w \in V$

$$w = \sum_{i=1}^{n} \langle w, v_i \rangle \, v_i.$$

Solution:

Let $w \in V$. Since $\{v_1, \ldots, v_n\}$ is a basis, it spans V and so $w = a_1 v_1 + \ldots + a_n v_n$ for some $a_1, \ldots, a_n \in F$. We need to show that $a_i = \langle w, v_i \rangle$ for each i. By definition of an inner product, we have

$$\langle w, v_i \rangle = \left\langle \sum_{j=1}^{n} a_j v_j, v_i \right\rangle = \sum_{j=1}^{n} a_j \langle v_j, v_i \rangle .$$

By definition of orthonormal we have $\langle v_j, v_i \rangle = 1$ if $i = j$ and $\langle v_i, v_j \rangle = 0$ if $i \neq j$. It then follows that the right side above $= a_i$ as desired.

5. (10 points) Consider the the vector space $\mathbb{R}^4$ equipped with the standard inner product. Find an orthogonal basis (you do not need to normalize it) for the subspace $W = \{(x_1, x_2, x_3, x_4) : x_1 + x_2 + x_3 - x_4 = 0\}$. You may omit the proof that it is an orthogonal basis.

Solution:

We start by finding a basis for W, clearly $\beta = \{(1,0,0,1),(0,1,0,1),(0,0,1,1)\}$ works. We use the Gram-Schmidt process to orthogonalize. Letting $v_1 = (1,0,0,1)$ we have

$$\begin{aligned} v_2 &= (0,1,0,1) - \frac{1}{2}\langle (0,1,0,1), v_1\rangle v_1 = (-1/2,1,0,1/2) \\ &= (0,1,0,1) + (-1/2,0,0,-1/2) \\ &= (-1/2,1,0,1/2) \end{aligned}$$

Finally

$$\begin{aligned} v_3 &= (0,0,1,1) - \frac{1}{2}\langle (0,0,1,1), v_1\rangle v_1 - 2/3\langle (0,0,1,1), v_2\rangle v_2 \\ &= (0,0,1,1) + (-1/2,0,0,-1/2) + (1/6,-1/3,0,-1/6) \\ &= (-1/3,-1/3,1,1/3). \end{aligned}$$

Then $\{v_1, v_2, v_3\}$ defined above are an orthogonal basis for W.

6. (10 points) Suppose that V is an inner product space, and that $v, w \in V$. Consider the linear transformation $T : V \to V$ defined $T(x) = \langle x, v \rangle w$. Calculate the adjoint of T.

Solution:

For every $x, y \in V$

$$\langle T(x), y \rangle = \langle \langle x, v \rangle w, y \rangle = \langle x, v \rangle \langle w, y \rangle = \left\langle x, \overline{\langle w, y \rangle} v \right\rangle = \langle x, \langle y, w \rangle v \rangle .$$

So, we have $T^*(y) = \langle y, w \rangle v$.

7. (10 points) Suppose that V is an inner product space, that $T : V \to V$ is a linear transformation, and that W is a subspace of V such that $T(W) \subset W$. Prove that $T^*(W^\perp) \subset W^\perp$.

Solution:

Suppose that $y \in T^*(W^\perp)$, we need to show that $\langle y, z\rangle = 0$ for every $z \in W$. Fix such a z. By definition of image, there is an $x \in W^\perp$ with $T^*(x) = y$. Since $(T^*)^* = T$

$$\langle y, z\rangle = \langle T^*(x), z\rangle = \langle x, T(z)\rangle \, .$$

Since $T(W) \subset W$ we have $T(z) \in W$ and so $\langle x, T(z)\rangle = 0$.

8. (10 points) Suppose that V is a two-dimensional inner product space, that β is an orthonormal basis for V, and that $T : V \to V$ is a linear transformation such that $[T]_\beta$ is upper triangular (in other words $([T]_\beta)_{2,1} = 0$), and such that T is normal. Show that $[T]_\beta$ is diagonal.

Solution:
Let $\{v_1, v_2\} = \beta$. To show that $[T]_\beta$ is diagonal, we need to show that $([T]_\beta)_{1,2} = 0$. Since β is an orthonormal basis for V, we have $([T]_\beta)_{1,2} = \langle T(v_2), v_1 \rangle = \langle v_2, T^*(v_1) \rangle$. Since $([T]_\beta)_{2,1} = 0$ we have that v_1 is an eigenvector for T with eigenvalue $([T]_\beta)_{1,1}$. From Theorem 6.15c and the fact that T is normal, we have that v_1 is an eigenvector for T^* with eigenvalue $\overline{([T]_\beta)_{1,1}}$ and so $\langle v_2, T^*(v_1) \rangle = \left\langle v_2, \overline{([T]_\beta)_{1,1}} v_1 \right\rangle = ([T]_\beta)_{1,1} \langle v_2, v_1 \rangle = 0$ where the last equation follows from the fact that β is orthogonal.

Extra Scratch Paper:

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