This week you will get practice drawing and understanding slope fields, making qualitative statements about solutions using them and some practice applying Euler’s method.

*Numbers in parentheses indicate the question has been taken from the textbook:
S. J. Schreiber, *Calculus for the Life Sciences*, Wiley,
and refer to the section and question number in the textbook.

1. (6.4.33) Consider the differential equation \( \frac{dy}{dt} = \frac{1}{t} \)
   (a) verify that \( y(t) = \ln t \) is a solution to this differential equation satisfying \( y(1) = 0 \).
   (b) Use Euler’s method to approximate \( y(2) = \ln 2 \) with \( h = 0.5 \).

2. (6.5) Draw phase lines, classify the equilibria, and sketch a solution satisfying the specified initial value for the equations in the following.
   (a) (6.5-2) \( \frac{dy}{dt} = 2 - 3y, \ y(0) = 2 \)
   (b) (6.5-5) \( \frac{dy}{dt} = y(y - 10)(20 - y), \ y(0) = 9 \)
   (c) (6.5-6) \( \frac{dy}{dt} = y(y - 5)(25 - y), \ y(0) = 7 \)
   (d) (6.5-7) \( \frac{dy}{dt} = \sin y, \ y(0) = 0.1 \)
   (e) (6.5-10) \( \frac{dy}{dt} = y^3 - 4y, \ y(0) = 0.1 \)

3. (6.5-33) To account for the effect of a generalist predator (with a type II functional response) on a population, ecologists often write differential equations of the form

\[
\frac{dN}{dt} = 0.1N \left( 1 - \frac{N}{1,000} \right) - \frac{10N}{1+N}
\]

(a) Sketch the phase line for this system.
(b) Discuss how the fate of the population depends on its initial abundance.

*Hint: dont worry about what the first sentence means, you dont need to know where the differential equation comes from.*

4. (6.5-39) Consider a population of clonally reproducing individuals consisting of two genotypes, \( a \) and \( A \), with per capita growth rates, \( r_a \) and \( r_A \), respectively. If \( N_a \) and \( N_A \) denote the densities of genotypes \( a \) and \( A \), then

\[
\frac{dN_a}{dt} = r_a N_a \quad \frac{dN_A}{dt} = r_A N_A
\]

Also, let \( y = \frac{N_a}{N_a + N_A} \) be the fraction of individuals in the population that are genotype \( a \). Show that \( y \) satisfies

\[
\frac{dy}{dt} = (r_a - r_A)y(1 - y)
\]

5. (6.5-40) In the Hawk-Dove replicator equation

\[
\frac{dy}{dt} = \frac{y}{2}(1-y)(C(1-y) - V)
\]

if the value \( V > 0 \) is specified, then find the range of values of \( C \) (in terms of \( V \)) that will ensure a polymorphism exists (i.e., find conditions that ensure the existence of an equilibrium \( 0 < y^* < 1 \) that is stable).

*(Hint: you do not need to know anything about the Hawk-Dove Replicator - though it is very interesting! - all you need to know is that \( V \) is a constant and \( C \) is a parameter. A polymorphism is a stable equilibrium between zero and one.)*
6. (6.5-41) Production of pigments or other protein products of a cell may depend on the activation of a gene. Suppose a gene is \textit{autocatalytic} and produces a protein whose presence activates greater production of that protein. Let \( y \) denote the amount of the protein (say, micrograms) in the cell. A basic model for the rate of this self-activation as a function of \( y \) is

\[
A(y) = \frac{ay^b}{k^b + y^b} \text{ micrograms/minute}
\]

where \( a \) represents the maximal rate of protein production, \( k > 0 \) is a “half saturation” constant, and \( b \geq 1 \) corresponds to the number of protein molecules required to activate the gene. On the other hand, proteins in the cell are likely to degrade at a rate proportional to \( y \), say \( cy \). Putting these two components together, we get the following differential equation model of the protein concentration dynamics:

\[
\frac{dy}{dt} = \frac{ay^b}{k^b + y^b} - cy
\]

(a) Verify that \( \lim_{y \to \infty} A(y) = a \) and \( A(k) = a/2 \).

(b) Verify that \( y = 0 \) is an equilibrium for this model and determine under what conditions it is stable.

\textit{(Hint: the definition of autocatalytic is given in the question, it is a gene that produces a protein whose presence activate greater production of that protein.)}

7. (6.5-42) Consider the model of an autocatalytic gene in Problem 41 with \( b = 1, k > 0, a > 0, \) and \( c > 0 \).

(a) Sketch the phase line for this model when \( ck > a \).

(b) Sketch the phase line for this model when \( ck < a \).