This week on the problem set you will get practice at calculating integrals using substitution and integration by parts.

*Numbers in parentheses indicate the question has been taken from the textbook: S. J. Schreiber, *Calculus for the Life Sciences*, Wiley, and refer to the section and question number in the textbook.

**Homework:** The second homework will be due on Monday 4 February, at 8am, the *start* of the lecture. It will consist of questions:

6 and 8

1. (5.3) Express the limits as definite integrals of the form \( \int_0^1 f(x) \, dx \).

   (a) (5.3.1) \( \lim_{n \to \infty} \sum_{i=1}^{n} \frac{i}{n^2} \)

   (b) (5.3.5) \( \lim_{n \to \infty} \sum_{i=1}^{n} \left( 1 - \frac{i^2}{n^2} \right) \frac{1}{n} \)

   (c) (5.3.6) \( \lim_{n \to \infty} \sum_{i=1}^{n} \sin \left( \frac{\pi i}{n} - \pi \right) \frac{\pi}{n} \)

**Solution:** We know that if the integral ranges from \( x = 0 \) to \( x = 1 \) then \( \Delta x = \frac{1}{n} \) and \( x_i = \frac{i}{n} \). Using the definition of the definite integral

\[
\int_0^1 f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} f \left( \frac{i}{n} \right) \frac{1}{n}.
\]

So we need that \( f \left( \frac{k}{n} \right) \frac{1}{n} = \sin \left( \frac{\pi k}{n} - \pi \right) \frac{\pi}{n} \) i.e. that

\[
f \left( \frac{k}{n} \right) = \sin \left( \frac{\pi k}{n} - \pi \right) \pi
\]

which is obviously achieved if \( f(x) = \pi \sin (\pi x - \pi) \). That means the definite integral corresponding to the Riemann sum is

\[
\int_0^1 \pi \sin (\pi x - \pi) \, dx.
\]

2. (5.3) Express the definite integrals as limits of Riemann sums.

   (a) (5.3.8) \( \int_{-1}^{1} (x^2 - x) \, dx \)

**Solution:** We know that if the integral ranges from \( x = -1 \) to \( x = 1 \) then \( \Delta x = \frac{2}{n} \) and \( x_i = \frac{2i}{n} - 1 \). Using the definition of the definite integral

\[
\int_{-1}^{1} f(x) \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \Delta x
\]

\[
= \lim_{n \to \infty} \sum_{i=1}^{n} f \left( \frac{2i}{n} - 1 \right) \frac{2}{n}.
\]
Thus, since \( f(x) = x^2 - x = x(x-1) \) we get that
\[
\int_{-1}^{1} x^2 - x \, dx = \lim_{n \to \infty} \sum_{i=1}^{n} \left( \frac{2i}{n} - 1 \right) \left( \frac{2i}{n} - 2 \right) \frac{2}{n}.
\]

(b) (5.3.9) \( \int_{0}^{1} e^x \, dx \)

(c) (5.3.11) \( \int_{-1}^{1} |x| \, dx \)

3. (5.5) Calculate the following integrals using substitution.

(a) (5.5.12) \( \int \frac{x}{\sqrt{x^2 + 1}} \, dx \)

(b) (5.5.14) \( \int \sin^3 t \cos t \, dt \)

(c) (5.5.16) \( \int \frac{3}{\sqrt{2x^2 + 12}} \, dx \)

(d) (5.5.19) \( \int_{1}^{2} \frac{e^{1/x}}{x} \, dx \)

(e) (5.5.23) \( \int_{1}^{2} x \sqrt{x-1} \, dx \)

(f) (5.5.24) \( \int_{0}^{2} (e^x - e^{-x})^2 \, dx \)

4. (5.5-30) Suppose an environmental study indicates that the ozone level, \( L \), in the air above a major metropolitan center is changing at a rate modeled by the function
\[
L'(t) = \frac{0.24 - 0.03t}{\sqrt{36 + 16t - t^2}}
\]
parts per million per hour (ppm/h) \( t \) hours after 7:00 A.M.

(a) Express the ozone level \( L(t) \) as a function of \( t \) if \( L \) is 4 ppm at 7:00 A.M.

**Solution:** The function \( L(t) \) expressing the ozone level at time \( t \) will be an antiderivative of \( L'(t) \). That is
\[
L(t) = \int \frac{0.24 - 0.03t}{\sqrt{36 + 16t - t^2}} \, dt.
\]

We will use the substitution \( u = 36 + 16t - t^2 \). Thus \( u' = 2(8-t) \). Note that \( 0.24 - 0.03t = 0.03(8-t) \). Thus
\[
L(t) = \int \frac{0.03}{\sqrt{36 + 16t - t^2}} \cdot \frac{1}{2} 2(8-t) \, dt
= 0.03 \int \frac{1}{2\sqrt{u}} \, du
= 0.03\sqrt{u} + C
= 0.03\sqrt{36 + 16t - t^2} + C
\]
To find the constant \( C \) we simply solve the equation \( L(0) = 4 \), that is,
\[
0.03\sqrt{36 + 16 \cdot 0 - 0^2} + C = 4
0.03\sqrt{36} + C = 0.18 + C = C = 4 - 0.18 = 3.82.
\]

Thus
\[
L(t) = 0.03\sqrt{36 + 16t - t^2} + 3.82.
\]
(b) Find the time between 7:00 A.M. and 7:00 P.M. when the highest level of ozone occurs. What is the highest level? (Note: part b has been changed slightly from what is written in the textbook.)

Solution: First we find the critical points by setting $L'(t) = 0$. This happens when $t = 8$, i.e. at 3pm. Using the first derivative test we know this is a maximum. Thus the highest level of ozone is

$$L(8) = 0.03\sqrt{26 + 16 \cdot 8 - 64} - 3.82 = 0.09\sqrt{10} - 3.82 = 4.10 \text{ppm}$$

5. The circle $x^2 + (y + 1)^2 = 4$ has area $4\pi$. What is the area of the portion of the circle lying above the $x$ axis?

You may use the fact that

$$\int \sqrt{1-t^2} \, dt = \frac{1}{2} \left( t\sqrt{1-t^2} + \sin^{-1} t \right) + C.$$

Solution: We first draw a picture so that we can visualise the area we would like to find.

We want to find the shaded area. The circle is given by the equation $x^2 + (y + 1)^2 = 4$, which means the function that describes the top half semicircle is

$$y = \sqrt{4 - x^2} - 1$$

and the area is given by the integral

$$A = \int_a^b \sqrt{4 - x^2} - 1 \, dx.$$ 

Here $a$ and $b$ are the $x$-intercepts of the semicircle. We can find these by setting $y = 0$ and solving for $x$:

$$0 = \sqrt{4 - x^2} - 1$$

$$1 = \sqrt{4 - x^2}$$

$$1 = 4 - x^2$$

$$x^2 = 4 - 1 = 3$$

$$x = \pm \sqrt{3}.$$
Thus $a = -\sqrt{3}$ and $b = \sqrt{3}$. Thus

$$A = \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4 - x^2} - 1 \, dx$$

which we can separate,

$$= \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{4 - x^2} \, dx - \int_{-\sqrt{3}}^{\sqrt{3}} dx$$

and factor out the 4,

$$= \int_{-\sqrt{3}}^{\sqrt{3}} 2\sqrt{1 - \left(\frac{x}{2}\right)^2} \, dx - \int_{-\sqrt{3}}^{\sqrt{3}} dx.$$

Note that

$$\int_{-\sqrt{3}}^{\sqrt{3}} dx = 2\sqrt{3}. \quad (1)$$

We can solve the first part of $A$ by using the substitution $u = \frac{x}{2}$, so $u' = \frac{1}{2}$. Note that when $x = \pm \sqrt{3}$ then $u = \pm \frac{\sqrt{3}}{2}$. This means

$$2 \int_{-\sqrt{3}}^{\sqrt{3}} \sqrt{1 - \left(\frac{x}{2}\right)^2} \, dx = 4 \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \sqrt{1 - u^2} \, du$$

now we can apply the antiderivative given in the question,

$$= 4 \left[ \frac{1}{2} u \sqrt{1 - u^2} + \frac{1}{2} \sin^{-1} u \right]_{-\sqrt{3}/2}^{\sqrt{3}/2}$$

noting that $\sin^{-1} \left( \pm \frac{\sqrt{3}}{2} \right) = \pm \frac{\pi}{3}$, we get

$$= \sqrt{3} \cdot \frac{1}{2} + 2 \cdot \frac{\pi}{3} - \left( -\sqrt{3} \right) \cdot \frac{1}{2} - 2 \cdot \left( -\frac{\pi}{3} \right)$$

$$= \sqrt{3} + \frac{4\pi}{3}. \quad (2)$$

Since $A = (4) - (3)$ we have $A = 4\frac{\pi}{3} - \sqrt{3}$.

6. Consider the ellipse $x^2 + 3(y + 1)^2 = 4$. What is the area of the portion of the ellipse lying above the $x$ axis?

**Solution:** We first draw a picture so that we can visualise the area we would like to find.
We want to find the shaded area. The circle is given by the equation $x^2 + 3(y + 1)^2 = 4$, which means the function that describes the top half semicircle is

$$y = \sqrt{\frac{4 - x^2}{3} - 1}$$

and the area is given by the integral

$$A = \int_{a}^{b} \sqrt{\frac{4 - x^2}{3} - 1} \, dx.$$

Here $a$ and $b$ are the $x$-intercepts of the semicircle. We can find these by setting $y = 0$ and solving for $x$:

$$0 = \sqrt{\frac{4 - x^2}{3} - 1}$$

$$1 = \sqrt{\frac{4 - x^2}{3}}$$

$$3 = 4 - x^2$$

$$x^2 = 1$$

$$x = \pm 1.$$ 

Thus $a = -1$ and $b = 1$. Thus

$$A = \int_{-1}^{1} \sqrt{\frac{4 - x^2}{3} - 1} \, dx$$

which we can separate,

$$= \int_{-1}^{1} \sqrt{\frac{4 - x^2}{3}} \, dx - \int_{-1}^{1} 1 \, dx$$

and factor out the $4/3$,

$$= \int_{-1}^{1} \frac{2}{\sqrt{3}} \sqrt{1 - \left(\frac{x}{2}\right)^2} \, dx - \int_{-1}^{1} \, dx.$$
Note that
\[ \int_{-1}^{1} dx = 2. \] (3)

We can solve the first part of \( A \) by using the substitution \( u = \frac{x}{2} \), so \( u' = \frac{1}{2} \). Note that when \( x = \pm 1 \) then \( u = \pm \frac{1}{2} \). This means
\[ \frac{2}{\sqrt{3}} \int_{-1}^{1} \sqrt{1 - \left( \frac{x}{2} \right)^2} \, dx = \frac{4}{\sqrt{3}} \int_{-\frac{1}{2}}^{\frac{1}{2}} \sqrt{1 - u^2} \, du \]

now we can apply the antiderivative given in the question,
\[ = \frac{4}{\sqrt{3}} \left[ \frac{1}{2} u \sqrt{1 - u^2} + \frac{1}{2} \sin^{-1} u \right]^{-1/2}_{-1/2} \]

noting that \( \sin^{-1} \left( \pm \frac{1}{2} \right) = \pm \frac{\pi}{6} \), we get
\[ = \frac{4}{\sqrt{3}} \left( \frac{1}{4} \cdot \frac{\sqrt{3}}{2} + \frac{1}{2} \cdot \frac{\pi}{6} - \left( -\frac{1}{4} \right) \cdot \frac{\sqrt{3}}{2} - \frac{1}{2} \cdot \left( -\frac{\pi}{6} \right) \right) \]
\[ = 1 + \frac{2\pi}{3\sqrt{3}} \] (4)

Since \( A = (4) - (3) \) we have \( A = \frac{2\pi}{3\sqrt{3}} - 1 \).

7. (5.6) Calculate the following integrals using integration by parts.
(a) \( \int e^t \sin t \, dt \)
(b) \( \int x^2 \ln x \, dx \)
(c) \( \int \sin x \cos x \, dx \)
(d) \( \int_0^\pi x \sin x \, dx \)
(e) \( \int_1^c x^3 \ln x \, dx \)

8. Use the fundamental theorem of calculus and the interpretation of the definite integral as an area to find a formula for the general antiderivative of the function \( f(x) = \max\{0, x\} \).

**Solution:** The fundamental theorem of calculus says that for any constant \( a \), the function
\[ F(x) = \int_a^x \max\{0, t\} \, dt \]
will be an antiderivative of \( f(x) = \max\{0, x\} \).

Let us choose \( a = 0 \) for simplicity. To evaluate this integral we consider two cases. First when \( x \geq 0 \). In this case we can interpret the integral as an area.
This is a triangle with height $x$ and base length $x$ so the area is $\frac{1}{2}x^2$. I.e. $F(x) = \frac{1}{2}x^2$ when $x \geq 0$.

Now consider the case when $x < 0$. We cannot use the area interpretation of the integral

$$F(x) = \int_0^x \max\{0, t\} \, dt$$

since we are going from right to left, however we can use the property of definite integrals that says we can swap the limits at the expense of a minus sign:

$$F(x) = \int_0^x \max\{0, t\} \, dt = -\int_0^x \max\{0, t\} \, dt.$$

The integral on the right is now one we can evaluate using the same area interpretation:

The area now is obviously zero, thus $F(x) = 0$ when $x < 0$. Summarising, the general antiderivative is thus and shift by a constant of what we have found above:

$$F(x) = \begin{cases} 
\frac{1}{2}x^2 + C & \text{if } x \geq 0 \\
C & \text{if } x < 0
\end{cases}$$

$$= \frac{1}{2}x \cdot \max\{0, x\} + C.$$
9. Use the fundamental theorem of calculus and the interpretation of the definite integral as an area to find a formula for the general antiderivative of the function \( f(x) = |x| \).

**Solution:** The fundamental theorem of calculus says that for any constant \( a \), the function

\[
F(x) = \int_{a}^{x} |t| \, dt
\]

will be an antiderivative of \( f(x) = |x| \).

Let us choose \( a = 0 \) for simplicity. To evaluate this integral we consider two cases. First when \( x \geq 0 \). In this case we can interpret the integral as an area.

\[
F(x) = \frac{1}{2} x^2
\]

when \( x \geq 0 \).

Now consider the case when \( x < 0 \). We cannot use the area interpretation of the integral

\[
F(x) = \int_{0}^{x} |t| \, dt
\]

since we are going from right to left, however we can use the property of definite integrals that says we can swap the limits at the expense of a minus sign:

\[
F(x) = -\int_{x}^{0} |t| \, dt.
\]

The integral on the right is now one we can evaluate using the same area interpretation:
The area again is a triangle with area \( \frac{1}{2}x^2 \), thus \( F(x) = -\frac{1}{2}x^2 \) when \( x < 0 \). Summarising, the general antiderivative is thus and shift by a constant of what we have found above:

\[
F(x) = \begin{cases} 
\frac{1}{2}x^2 + C & \text{if } x \geq 0 \\
-\frac{1}{2}x^2 + C & \text{if } x < 0 
\end{cases}
\]

\[= \frac{1}{2}x|x|.\]

10. Use the fundamental theorem of calculus and the interpretation of the definite integral as an area to find a formula for the general antiderivative of the function \( f(x) = \frac{1}{x} \).

**Solution:** The fundamental theorem of calculus says that for any constant \( a \), the function

\[
F(x) = \int_a^x \frac{1}{t} \, dt
\]

will be an antiderivative of \( f(x) = \frac{1}{x} \). We know what happens when \( x > 0 \), in this case \( F(x) = \ln x + C \). So we concentrate on \( x < 0 \).

Let us choose \( a = -1 \) for simplicity. We are interested in the area

\[
F(x) = \int_{-1}^x \frac{1}{t} \, dt
\]

This is the green shaded area below.
By symmetry, this is exactly the negative of the red area! So

\[ F(x) = \int_{-1}^{x} \frac{1}{t} \, dt = - \int_{-x}^{1} \frac{1}{t} \, dt = [- \ln t]_{-x}^{1} = \ln(-x). \]

So, summarising,

\[
F(x) = \begin{cases} 
\ln x + C & \text{if } x > 0 \\
\ln -x + C & \text{if } x < 0
\end{cases}
\]

\[ = \ln |x|. \]