Midterm 1
UCLA: Math 3B, Winter 2019

Instructor: Noah White
Date: 28 January 2019

• This exam has 3 questions, for a total of 30 points.
• Please print your working and answers neatly.
• Write your solutions in the space provided showing working.
• Indicate your final answer clearly.
• You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
• Non programmable and non graphing calculators are allowed.

Name: 
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Discussion section (please circle):

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Question 1 is multiple choice. Indicate your answers in the table below. The following three pages will not be graded, your answers must be indicated here.

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☐ I wish to opt out of having my exam graded using Gradescope.
1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (1 point) The function \( f(x) = e^x(1 + e^x)^{-1} \) has
   A. a horizontal asymptote at \( y = 0 \).
   B. a horizontal asymptote at \( y = 1 \) in both directions.
   C. no asymptotes.
   D. a vertical asymptote at \( x = 0 \).

(b) (1 point) The function \( g(x) = xe^{-x} \) has a critical point at
   A. \( x = e^{-2} \).
   B. \( x = 1 \).
   C. \( x = -1 \).
   D. \( x = 0 \).

(c) (1 point) The function \( f(x) = e^{5 + 4x - x^2} \) has a
   A. local maximum at \( x = 2 \).
   B. local minimum at \( x = 2 \).
   C. local maximum at \( x = 1 \).
   D. local minimum at \( x = 1 \).
(d) (2 points) An antiderivative of \( f(t) = t \cos t \) is given by
A. \( \frac{1}{2} t^2 \cos t + t \sin t \)
B. \( t \sin t \)
C. \( \sin t^2 \)
D. \( t \sin t + \cos t \)

(e) (2 points) The function \( x e^x - 1 \) has a
A. vertical asymptote at \( x = 0 \).
B. horizontal asymptote at \( y = 1 \).
C. no vertical asymptotes.
D. no horizontal asymptotes.

(f) (2 points) An antiderivative of \( h(x) = \frac{1 + \ln x}{\sqrt{x \ln x}} \) is given by
A. \( -\ln(\sqrt{x \ln x}) \)
B. \( 1 - 2\sqrt{x \ln x} \)
C. \( \sqrt{x \ln x} \)
D. \( 1 + \ln x \)
2. Let \( f(x) = \frac{x}{x^2 - 1} \). Note that \( f'(x) = -\frac{x^2 + 1}{(x^2 - 1)^2} \) and \( f''(x) = \frac{2x(x^2 + 3)}{(x^2 - 1)^3} \).

(a) (2 points) Find the \( x \) and \( y \) intercepts of \( f(x) \).

**Solution:** \( x \)-intercept and \( y \)-intercept at the origin.

(b) (1 point) Does \( f(x) \) have any horizontal asymptotes? If so what are they?

**Solution:** We need to evaluate

\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{\sqrt{x^4}}{x^2 - 1} = 0.
\]

and

\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{\sqrt{x^4}}{x^2 - 1} = 0.
\]

So horizontal asymptotes at \( y = 0 \) in both directions.

(c) (1 point) Does \( f(x) \) have any vertical asymptotes? If so what are they?

**Solution:** The denominator of \( f(x) \) can be zero when \( x = \pm 1 \). To confirm these are both vertical asymptotes we evaluate the following limits.

\[
\lim_{x \to 1^+} \frac{\sqrt{x}}{x^2 - 1} = \infty
\]

\[
\lim_{x \to 1^-} \frac{\sqrt{x}}{x^2 - 1} = -\infty
\]

\[
\lim_{x \to -1^+} \frac{\sqrt{x}}{x^2 - 1} = -\infty
\]

\[
\lim_{x \to -1^-} \frac{\sqrt{x}}{x^2 - 1} = \infty
\]

(d) (2 points) For what \( x \) is the first derivative \( f'(x) \) positive?

**Solution:** Both the numerator and the denominator are always positive so the derivative is always negative.
(e) (2 points) For what $x$ is the second derivative $f''(x)$ positive?

**Solution:** The denominator can be zero when $x = \pm 1$, the numerator when $x = 0$. By testing $f''(-2) < 0, f''(-0.5) > 0, f''(0.5) < 0, f''(2) > 0$ we see that the second derivative is positive when $-1 < x < 0$ and $x > 1$.

(f) (3 points) On the graph provided, sketch $f(x)$
3. You are stranded on an island. You observe a boat 10 km directly to the West. The boat is moving in a straight line at a constant speed. One hour later, you observe the ship to be 1 km East and 2 km North of its original position.

*Hint: to make the question slightly more concrete, let’s plot everything on the plane. If the boat’s original position is (0,0), then the island is at (10,0) and after one hour the boat will be at (1,2).*

(a) (2 points) Draw a picture representing the position of the island and the trajectory of the boat.

(b) (8 points) How far into the future will the boat be closest to the island?

**Solution:**  
Every hour, the boat moves 1 km east and 2 km north. Thus after $t$ hours, the boat will be at position $(t, 2t)$. We want to minimise the distance between this point and the island, i.e. the point $(10, 0)$. The distance is given by  

$$d(t) = \sqrt{(t - 10)^2 + (2t)^2} = \sqrt{5t^2 - 20t + 100}.$$  

We also note that a reasonable domain for this function is $t \in [0, \infty)$. To find the minimum we first find the critical points. The derivative is  

$$d'(t) = \frac{5t - 10}{\sqrt{5t^2 - 20t + 100}}.$$  

This is never undefined since the denominator is never zero: you can see this in two different ways, first it is a distance which is obviously never zero from the picture, and secondly one can use the discriminant of this quadratic and see it is negative. Setting it equal to zero gives the only critical point at $t = 2$.

Now to find the global min we look at the value of $d(t)$ at $t = 2$ and at the end point $t = 0$ and compare these to the limit $\lim_{t \to \infty} d(t)$.

$$d(0) = 10, \quad d(2) = \sqrt{80} < 10, \quad \lim_{t \to \infty} \sqrt{5t^2 - 20t + 100} = \infty.$$  

Thus $t = 2$ is indeed a global minimum.
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