Instructor: Noah White
Date:

- This exam has 3 questions, for a total of 36 points.
- Please print your working and answers neatly.
- Write your solutions in the space provided showing working.
- Indicate your final answer clearly.
- You may write on the reverse of a page or on the blank pages found at the back of the booklet however these will not be graded unless very clearly indicated.
- Non programmable and non graphing calculators are allowed.

Name: 

ID number: 

Discussion section (please circle):

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<th>Day/TA</th>
<th>Louis</th>
<th>Matthew</th>
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<td>Tuesday</td>
<td>1A</td>
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<td>Thursday</td>
<td>1B</td>
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Question 1 is multiple choice. Once you are satisfied with your solutions, indicate your answers by marking the corresponding box in the table below.

*Please note! The following three pages will not be graded. You must indicate your answers here for them to be graded!*

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<thead>
<tr>
<th>Part</th>
<th>A</th>
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1. Each of the following questions has exactly one correct answer. Choose from the four options presented in each case. No partial points will be given.

(a) (2 points) The function $f(x) = \frac{1}{1+x^2}$ has
\begin{itemize}
  \item A. a vertical asymptote at $x = -1$.
  \item B. a horizontal asymptote at $y = 0$.
  \item C. no asymptotes.
  \item D. a slanted asymptote.
\end{itemize}

(b) (2 points) The function $g(x) = x - \sin x$ has a critical point at
\begin{itemize}
  \item A. $x = \pi/2$.
  \item B. $x = 2$.
  \item C. $x = \pi$.
  \item D. $x = 0$.
\end{itemize}

(c) (2 points) The function $f(x) = \frac{1}{5-4x+x^2}$ has a
\begin{itemize}
  \item A. local minimum at $x = 2$.
  \item B. local maximum at $x = 2$.
  \item C. local minimum at $x = 1$.
  \item D. local maximum at $x = 1$.
(d) (2 points) An antiderivative of \( h(t) = 2e^{2x} - 4x \) is given by

A. \( 2x^2 - \cos x^2 \)
B. \( 2x^2 - 2e^{2x} \)
C. \( e^{2x} - 2x^2 + \frac{5}{4} \)
D. \( 4e^{2x} - 4 \)

(e) (2 points) The area \( \int_{2}^{3} \ln x \, dx \) can be expressed as the limit as \( n \to \infty \) of

A. \( \sum_{k=1}^{n} \ln \left( 2 + \frac{k}{n} \right) \)
B. \( \sum_{k=1}^{n} \frac{2}{n} + \frac{k}{n^2} \)
C. \( \frac{1}{n} \sum_{k=1}^{n} \left[ \ln (2n + k) - \ln n \right] \)
D. \( \sum_{k=1}^{n} \frac{k}{n^2} \)

(f) (2 points) Evaluate the definite integral \( \int_{0}^{\pi} \cos(x - \frac{\pi}{2}) \, dx \)

A. 1
B. 2
C. \( \pi \)
D. 0
2. Let \( f(x) = \frac{1}{1+e^{2x}} \). Note that \( f'(x) = \frac{-2e^{2x}}{(1+e^{2x})^2} \) and \( f''(x) = \frac{-4e^{2x}(1-e^{2x})}{(1+e^{2x})^3} \).

(a) (2 points) Find the \( x \) and \( y \) intercepts of \( f(x) \).

**Solution:** No \( x \)-intercepts, \( y \)-intercept at \( y = 0.5 \).

(b) (2 points) Does \( f(x) \) have any horizontal asymptotes? If so what are they?

**Solution:** We need to evaluate
\[
\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \frac{1}{1 + e^{2x}} = 0.
\]
and
\[
\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{1}{1 + e^{2x}} = 1.
\]
So we have horizontal asymptote in the positive direction at \( y = 0 \) and in the negative direction at \( y = 1 \).

(c) (1 point) Does \( f(x) \) have any vertical asymptotes? If so what are they?

**Solution:** The denominator of \( f(x) \) can never be zero so the function is always well defined, thus no vertical asymptotes.

(d) (2 points) For what \( x \) is the first derivative \( f'(x) \) positive?

**Solution:** The denominator is always positive, and the numerator is always negative. Thus \( f'(x) \) is negative for all \( x \).
(e) (2 points) For what $x$ is the second derivative $f''(x)$ positive?

**Solution:** Again, the denominator is always positive. The numerator is positive when $e^{2x} > 1$

i.e. when $2x > 0$ thus when $x > 0$.

(f) (3 points) On the graph provided, sketch $f(x)$
3. Two boats are travelling to and from an island in straight lines, as indicated below. Boat $A$ is heading due east at a constant speed of $1 \text{ m/h}$ and at time $t = 0$ is $3$ miles from the island. Boat $B$ is heading due south at $2 \text{ m/h}$ and at time $t = 0$ is at the island. Both boats stop travelling after boat $A$ reaches the island.

(a) (5 points) Write down an expression for the distance $s(t)$ between the boats after $t$ hours have elapsed.

**Solution:** After $t$ hours, the distance from $A$ to the island is $3 - t$ miles and the distance from $B$ to the island is $2t$ miles. Thus the distance between them (using Pythagoras’ theorem) is

$$d(t) = \sqrt{(3 - t)^2 + 4t^2} = \sqrt{9 - 6t + 5t^2}.$$ 

(b) (2 points) What is a sensible domain for $s(t)$?

**Solution:** The domain for this function is $[0, 3]$ (since otherwise $A$ reaches the island). Answers like $[0, \infty)$ were also accepted.
(c) (5 points) At what point in time, are the boats closest together?

**Solution:** The derivative is

\[ d'(t) = \frac{-3 + 5t}{\sqrt{9 - 6t + 5t^2}} \]

which is only zero when \( t = \frac{3}{5} \). It is defined everywhere since the quadratic on the bottom never vanishes (this is easy to see since \( 6^2 - 4 \cdot 9 \cdot 5 < 0 \), or from the physical situation since it is obvious that the boats will never be in the same place.)

Thus we only have a single critical point \( t = 3/5 \) on our closed interval, so using the closed interval method we just need to compare the values

\[ d(0) = 3, \quad d(3/5) = \frac{6}{\sqrt{5}}, \quad d(3) = 6. \]

Since \( 6/\sqrt{3} \) is the smallest value (easily seen using a calculator or noticing that \( \sqrt{5} > 2 \) so \( 6/\sqrt{5} < 6/2 = 3 \)) we can conclude that \( t = 3/5 \) is the minimum (i.e. 36 minutes after starting they are closest together).
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