Often it is impossible to solve a differential equation. E.g.

\[ \frac{dy}{dt} = y^2 + t \]

(the *Riccati equation*) has no solutions that can be written in terms of usual functions like \( \sin x \), \( e^x \), etc.
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(the \textit{Riccati equation}) has no solutions that can be written in terms of usual functions like \( \sin x \), \( e^x \), etc.

We want a method to estimate \( y(t) \) is we know that \( y(t_0) = y_0 \).
Euler's method

Let’s use Euler’s method!
Idea behind Euler's method

Suppose $y(t)$ is a solution to

$$\frac{dy}{dt} = f(t, y)$$

and that $y(t_0) = y_0$. 

If $h$ is a small number (e.g. $h = 0.1$), then we approximate $y(t_0 + h)$ using $(t_0, y_0)$ and $(t_0 + h, y_1)$.
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$$(t_0, y_0) \quad \quad (t_0 + h, y_1)$$

with

$$y_1 - y_0 \approx h f(t_0, y_0).$$
Idea behind Euler's method

\[ y(t_0 + h) \approx y_1 \]
Idea behind Euler's method

\[ y'(t_0) = \frac{\text{rise}}{\text{run}} = \frac{y_1 - y_0}{h} \]
Idea behind Euler's method

\[ y' = f(t, y) \]

\[
(t_0, y_0) \\
(h, y_1 - y_0) \\
(t_0 + h, y_1)
\]

\[
y(t_0 + h) \approx y_1 = y_0 + hy'(t_0) = y_0 + hf(t_0, y_0)
\]
Idea behind Euler's method

\[ \frac{dy}{dt} = f(t, y) \]

If we know that the solution satisfies \( y(t_0) = y_0 \) then

- let \( h \) be a small step forward in time

\[ y_1 \approx y(t_1) = y_0 + hf(t_0, y_0) \]
Idea behind Euler's method

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\frac{dy}{dt} = f(t, y)
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If we know that the solution satisfies \(y(t_0) = y_0\) then

- let \(h\) be a small step forward in time
- we can get an approximate value for the solution at \(t = t_0 + h = t_1\)
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\[ \frac{dy}{dt} = f(t, y) \]

If we know that the solution satisfies \( y(t_0) = y_0 \) then

- let \( h \) be a small step forward in time
- we can get an approximate value for the solution at \( t = t_0 + h = t_1 \)
- i.e. \( y(t_1) \approx y_1 \) where

\[ y_1 = y_0 + hf(t_0, y_0) \]
Eulers method

To carry out Eulers method, we simply repeat this a number of times!

\[
\frac{dy}{dt} = f(t, y)
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Given an initial value \( y(t_0) = y_0 \). To approximate \( y(t) \) at \( t = a \) follow the steps:
- Choose an increment \( h \)
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Given an initial value \( y(t_0) = y_0 \). To approximate \( y(t) \) at \( t = a \) follow the steps:

- Choose an increment \( h \)
- Set \( t_1 = t_0 + h \)
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- set \( y_2 = y_1 + hf(t_1, y_1) \)
- keep repeating until \( t_n \approx a \)
- then \( y(a) \approx y_n \).
An example

We will approximate $y(2)$, where $y$ obeys

$$\frac{dy}{dt} = y^2 + t$$

and $y(0) = 0$. Let $h = 0.5$.

<table>
<thead>
<tr>
<th>Iter.</th>
<th>$x$</th>
<th>$y$</th>
</tr>
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<tbody>
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<td>0</td>
<td>0</td>
</tr>
<tr>
<td>1</td>
<td>0</td>
<td>$0 + 0.5 \cdot (0^2 + 0)$</td>
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<tr>
<td>2</td>
<td>0</td>
<td>$0 + 0.5 \cdot (0^2 + 0)$</td>
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<tr>
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<td>0</td>
<td>$0 + 0.5 \cdot (0^2 + 0)$</td>
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<tr>
<td>4</td>
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<td>$0 + 0.5 \cdot (0^2 + 0)$</td>
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</tr>
<tr>
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<td>0</td>
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<td>2</td>
<td>1.0</td>
<td>0.25</td>
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<td>$y_2 = 0 + 0.5 \cdot (0^2 + 0.5)$</td>
<td>$y_3 = 0.25 + 0.5 \cdot (0.25^2 + 1)$</td>
</tr>
<tr>
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<td>0.25</td>
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<td>0.78</td>
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$y_1 = 0 + 0.5 \cdot (0^2 + 0)$

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We will approximate \( y(2) \), where \( y \) obeys

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<th>( y_4 = 0.78 + 0.5 \cdot (0.78^2 + 1.5) )</th>
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<td>0.78</td>
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