Fractal uncertainty principles for ellipsephic sets

Nicholas Hu

The University of California, Los Angeles (UCLA)

May 27, 2022

Introduction

The fractal uncertainty principle (Dyatlov–Zahl, 2016)

"*No function can be localized in both position and frequency close to a fractal set.*"

- Applications to quantum chaos (eigenfunction control and spectral gaps on hyperbolic surfaces).
- Connections to harmonic analysis (additive energy, Fourier decay, and Fourier restriction estimates; additive combinatorics; spectral sets).

Continuous uncertainty principles

Let $\mathcal{F}_h : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ ($0 < h \ll 1$) be the unitary semiclassical Fourier transform

$$\mathcal{F}_h f(\xi) := \frac{1}{\sqrt{2\pi h}} \int_{\mathbb{R}} e^{-ix\xi/h} f(x)\, dx.$$

Continuous uncertainty principles (Dyatlov–Zahl, 2016)

An h-dependent family of sets $\{X_h\}_{h>0} \subseteq \mathcal{P}(\mathbb{R})$ is said to satisfy an **uncertainty principle** with **exponent** $\beta \in \mathbb{R}$ if

$$\|\mathbb{1}_{X_h} \mathcal{F}_h \mathbb{1}_{X_h}\|_{L^2(\mathbb{R}) \to L^2(\mathbb{R})} = \mathcal{O}(h^\beta) \quad \text{as } h \to 0.$$

(The subscript on X is typically elided.)

Continuous uncertainty principles

Example: $X = [0, h]$.

By Hölder's inequality,

$$\begin{aligned} \|\mathbb{1}_X \mathcal{F}_h \mathbb{1}_X\|_{L^2 \to L^2} &\le \|\mathbb{1}_{[0,h]}\|_{L^\infty \to L^2} \|\mathcal{F}_h\|_{L^1 \to L^\infty} \|\mathbb{1}_{[0,h]}\|_{L^2 \to L^1} \\ &= h^{1/2} \cdot (2\pi h)^{-1/2} \cdot h^{1/2}, \end{aligned}$$

so X satisfies an uncertainty principle with exponent $\frac{1}{2}$.

Continuous fractal uncertainty principles

For "regular" fractal sets $X \subseteq [0,1]$ of "dimension" $\delta \in [0,1]$, we have the **basic fractal uncertainty principle (FUP) exponent**

$$\beta_0 := \max\left\{0, \frac{1}{2} - \delta\right\}.$$

Can this be improved upon (by obtaining $\beta > \beta_0$ for δ-regular families of sets)?

- Yes – when $\delta < 1$, we can obtain $\beta > 0$: improvement for $\delta \geq \frac{1}{2}$ (Bourgain–Dyatlov, 2017).
- Yes – when $\delta > 0$, we can obtain $\beta > \frac{1}{2} - \delta$: improvement for $\delta \leq \frac{1}{2}$ (Dyatlov–Jin, 2018).

Ellipsephic sets

An **ellipsephic** ([ˌɪ.lɪp'sɛf.ɪk]) **set** in **base** M is a set consisting of all k-digit integers in base M with digits in some nonempty **alphabet** $\mathcal{A} \subseteq \mathbb{Z}_M := \{0, 1, \ldots, M-1\}$. Such a set is denoted $\mathcal{C}_k(M, \mathcal{A})$ (or simply $\mathcal{C}_k$). In other words,

$$\mathcal{C}_k = \mathcal{C}_k(M, \mathcal{A}) := \left\{ \sum_{d=0}^{k-1} a_d M^d : a_d \in \mathcal{A} \right\}.$$

Note that $\mathcal{C}_k \subseteq \mathbb{Z}_N$ for $N := M^k$ and $|\mathcal{C}_k| = |\mathcal{A}|^k = N^{\log_M |\mathcal{A}|}$.

The **dimension** of $\mathcal{C}_k(M, \mathcal{A})$ is $\delta := \log_M |\mathcal{A}| \in [0, 1]$.
We will not consider trivial alphabets with $\delta = 0$ ($|\mathcal{A}| = 1$) or $\delta = 1$ ($|\mathcal{A}| = M$).

Ellipsephic sets

Example: $M = 10$, $\mathcal{A} = \{2, 7\}$.

$$\mathcal{C}_2(M, \mathcal{A}) = \{22, 27, 72, 77\}$$
$$\delta = \log_{10} 2 \approx 0.3$$

Discrete fractal uncertainty principles

Let $\mathcal{F}_N : \mathbb{C}^N \to \mathbb{C}^N$ be the unitary discrete Fourier transform

$$\mathcal{F}_N u(j) := \frac{1}{\sqrt{N}} \sum_{\ell \in \mathbb{Z}_N} e^{-2\pi i j\ell/N} u(\ell) = \frac{1}{\sqrt{N}} \sum_{\ell \in \mathbb{Z}_N} \omega_N^{j\ell} u(\ell).$$

Discrete fractal uncertainty principles (Dyatlov–Jin, 2017)

A family of ellipsephic sets $\{\mathcal{C}_k(M, \mathcal{A})\}_{k \geq 1}$ is said to satisfy an **uncertainty principle** with **exponent** $\beta \in \mathbb{R}$ if

$$\|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}\|_{\ell^2(\mathbb{Z}_N) \to \ell^2(\mathbb{Z}_N)} \lesssim_{M,\mathcal{A}} N^{-\beta}.$$

Discrete fractal uncertainty principles

Example: $M = 10$, $\mathcal{A} = \{2, 7\}$, $k = 1$.

$$\|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}\|_2 = \|\mathbb{1}_{\{2,7\}} \mathcal{F}_{10} \mathbb{1}_{\{2,7\}}\|_2 = \left\| \frac{1}{\sqrt{10}} \begin{bmatrix} \omega_{10}^{2\cdot 2} & \omega_{10}^{2\cdot 7} \\ \omega_{10}^{7\cdot 2} & \omega_{10}^{7\cdot 7} \end{bmatrix} \right\|_2$$

Example: $M = 10$, $\mathcal{A} = \{0, 5\}$, $k = 1$.

$$\|\mathbb{1}_{\mathcal{C}_k} \mathcal{F}_N \mathbb{1}_{\mathcal{C}_k}\|_2 = \left\| \frac{1}{\sqrt{10}} \begin{bmatrix} \omega_{10}^{0\cdot 0} & \omega_{10}^{0\cdot 5} \\ \omega_{10}^{5\cdot 0} & \omega_{10}^{5\cdot 5} \end{bmatrix} \right\|_2 = \left\| \frac{1}{\sqrt{10}} \begin{bmatrix} \omega_{10}^{2\cdot 2} & \omega_{10}^{2\cdot 7} \\ \omega_{10}^{7\cdot 2} & \omega_{10}^{7\cdot 7} \end{bmatrix} \right\|_2$$

Notice that $\{0, 5\} + 2 = \{2, 7\}$ and

$$\begin{bmatrix} \omega_{10}^{0\cdot 2} & \\ & \omega_{10}^{5\cdot 2} \end{bmatrix} \begin{bmatrix} \omega_{10}^{0\cdot 0} & \omega_{10}^{0\cdot 5} \\ \omega_{10}^{5\cdot 0} & \omega_{10}^{5\cdot 5} \end{bmatrix} \begin{bmatrix} \omega_{10}^{2\cdot 0} & \\ & \omega_{10}^{2\cdot 5} \end{bmatrix} \begin{bmatrix} \omega_{10}^{2\cdot 2} & \\ & \omega_{10}^{2\cdot 2} \end{bmatrix} = \begin{bmatrix} \omega_{10}^{2\cdot 2} & \omega_{10}^{2\cdot 7} \\ \omega_{10}^{7\cdot 2} & \omega_{10}^{7\cdot 7} \end{bmatrix}.$$

Discrete fractal uncertainty principles

For ellipsephic sets of dimension $\delta \in [0, 1]$, we have the **basic FUP exponent**

$$\beta_0 := \max\left\{0, \frac{1}{2} - \delta\right\}.$$

Can this be improved upon (by obtaining $\beta > \beta_0$ for ellipsephic sets of dimension δ)?

- Yes – for all $0 < \delta < 1$, we can obtain $\beta > \beta_0$ (Dyatlov–Jin, 2017).

Discrete fractal uncertainty principles

Proof (basic FUP exponent):

$$\|\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}\|_2 \le \|\mathcal{F}_N\|_2 = 1 = N^{-0}$$

$$\|\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}\|_2 \le \|\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}\|_{\mathrm{F}} = \sqrt{|\mathcal{C}_k|^2\left(\frac{1}{\sqrt{N}}\right)^2} = N^{-\left(\frac{1}{2}-\delta\right)}$$

Discrete fractal uncertainty principles

Let $r_k = r_k(M, \mathcal{A}) := \|\mathbb{1}_{\mathcal{C}_k(M,\mathcal{A})}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k(M,\mathcal{A})}\|_2$.

- Upper bound:
$$\beta \leq \frac{1-\delta}{2}.$$
  - Apply $\mathbb{1}_{\mathcal{C}_k}\mathcal{F}_N\mathbb{1}_{\mathcal{C}_k}$ to $1_{\{x\}}$ for some $x \in \mathcal{C}_k$.
- Alphabet shift: if $a \in \mathbb{Z}_M$ and $\mathcal{A} \subseteq \{0, 1, \ldots, (M-1) - a\}$, then
$$r_k(M, \mathcal{A} + a) = r_k(M, \mathcal{A}).$$
  - Notice that $\mathcal{C}_k(M, \mathcal{A} + a) = \mathcal{C}_k(M, \mathcal{A}) + (a \cdots a)_M$ and apply the shift theorem for the DFT.
- Submultiplicativity:
$$r_{k_1+k_2} \leq r_{k_1} r_{k_2}.$$
  - Notice that $\mathcal{C}_{k_1+k_2} = \mathcal{C}_{k_1}\mathcal{C}_{k_2}$ (in the sense of *concatenation*) and use an FFT-like decomposition.

Discrete fractal uncertainty principles

Let $\beta_k = -\log_N r_k = -\dfrac{\log_M r_k}{k}$.

Fekete's lemma applied to the subadditive sequence $\{\log_M r_k\}_{k\geq 1}$ allows us to compute the maximal β as

$$\beta = \lim_{k\to\infty} \beta_k = \sup_{k\geq 1} \beta_k.$$

Recent work

How does (the maximal) β depend on $(M, \mathcal{A})$?

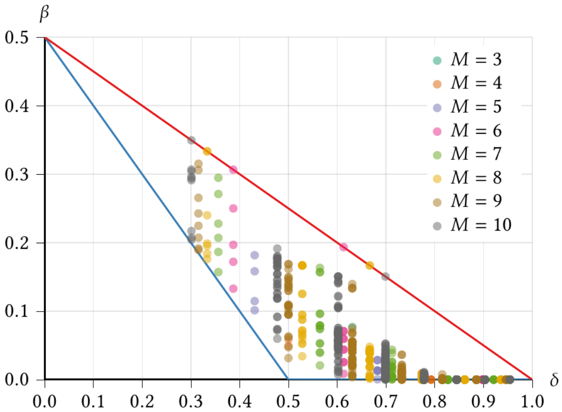

Figure: Numerically approximated FUP exponents for all alphabets with $M \leq 10$.

Recent work

For any $\delta \leq \frac{1}{2}$, the improvement over the basic exponent can be *arbitrarily small*, in that there exist sequences $\{(M_j, \mathcal{A}_j)\}$ with $\delta(M_j, \mathcal{A}_j) \to \delta$ and $\beta(M_j, \mathcal{A}_j) \to \beta_0$ (Dyatlov–Jin, 2017).

Is this also true for $\delta > \frac{1}{2}$? (Dyatlov, 2019)

- Yes (· , 2021).
- For some sequences, the improvement over the basic exponent might even be (nearly) *exponentially small*. (We have an upper bound for β_1 so far.)

Recent work

Which bases/alphabets attain the upper bound $\beta = \frac{1-\delta}{2}$? (Dyatlov–Jin, 2017)

- 'Spectral' alphabets (Dyatlov–Jin, 2017).
- Numerical experiments ($M \leq 25$; later, $M \leq 39$) suggest that these might be the only ones.

Thank you for your attention!